Pavol Quittner
Philippe Souplet

Superlinear Parabolic Problems

Blow-up, Global Existence and Steady States

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Authors:

Pavol Quittner
Department of Applied Mathematics
and Statistics
Comenius University
Mlynská dolina
84248 Bratislava
Slovakia
e-mail: quittner@fmph.uniba.sk

Philippe Souplet
Laboratoire Analyse Géométrie et Applications
UMR CNRS 7539 - Institut Galilée
Université Paris-Nord
99 av. J.-B. Clément
93430 Villetaneuse
France
souplet@math.univ-paris13.fr

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Introduction

This book is devoted to the qualitative study of solutions of superlinear elliptic and parabolic partial differential equations and systems. Here "superlinear" means that the problems involve nondissipative terms which grow faster than linearly for large values of the solutions. This class of problems contains, in particular, a number of reaction-diffusion systems which arise in various mathematical models, especially in chemistry, physics and biology.

For parabolic problems of this type it is known that a solution may cease to exist in a finite time as a consequence of its L^∞-norm becoming unbounded: The solution blows up. On the other hand, in many of these problems there exist also global solutions (in particular, stationary solutions). Both global and blowing-up solutions may be very unstable and they may exhibit a rather complicated asymptotic behavior.

Concerning elliptic problems, we consider questions of existence and nonexistence, multiplicity, regularity, singularities and a priori estimates. Special emphasis is put on those results which are useful in the investigation of the corresponding parabolic problems. As for parabolic problems, we study the questions of local and global existence, a priori estimates and universal bounds, blow-up, asymptotic behavior of global and nonglobal solutions.

The study of superlinear parabolic and elliptic equations and systems has attracted the attention of many mathematicians during the past decades. Although a lot of challenging problems have already been solved, there are still many open questions even in the case of the simplest possible model problems. Unfortunately, most of the material, including many of the fundamental ideas, is scattered throughout hundreds of research articles which are not always easily readable for non-specialists. One of the main purposes of this book is thus to give an up-to-date and, as much as possible, self-contained account of the most important results and ideas of the field. In particular we try to find a balance between fundamental ideas and current research. Special effort is made to describe in a pedagogical way the main methods and techniques used in the study of these problems and to clarify the connections between several important results. Moreover, a number of the original proofs have been significantly simplified. In this way, the topic should be accessible to a larger audience of non-specialists.

The book contains five chapters. The first two are intended to be an introduction to the field and to enable the reader to get acquainted with the main ideas by studying simple model problems, respectively of elliptic and parabolic type. These model problems are of the form

$$\left.\begin{aligned} -\Delta u &= f(u), && x \in \Omega, \\ u &= 0, && x \in \partial\Omega, \end{aligned}\right\} \tag{0.1}$$

and

$$\left.\begin{aligned} u_t - \Delta u = f(u), \qquad & x \in \Omega,\ t > 0, \\ u = 0, \qquad & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \qquad & x \in \Omega, \end{aligned}\right\} \tag{0.2}$$

where $\Omega \subset \mathbb{R}^n$ and f is a superlinear function, typically $f(u) = |u|^{p-1}u$ for some $p > 1$. The subsequent three chapters are devoted to problems with more complex structure; namely, elliptic and parabolic systems, equations with gradient depending nonlinearities, and nonlocal equations. They include several problems arising in biological or physical contexts. These chapters contain many developments which reflect several aspects of current research. Although the techniques introduced in Chapters I and II provide efficient tools to attack some aspects of these problems, they often display new phenomena and specifically different behaviors, whose study requires new ideas. Many open problems are mentioned and commented.

For the reader's convenience we have collected a number of frequently used results in several appendices. These include estimates of solutions of linear elliptic and parabolic equations, maximum principles, and basic notions from dynamical systems. Also, in one of the appendices, we give an account of the local theory of semilinear parabolic problems based on the abstract framework of interpolation-extrapolation spaces. However, this material is not essential for the understanding of the main contents of the book and can be left for a second reading. In particular, for the case of the model problem (0.2), the most useful results on local existence-uniqueness are proved by more elementary methods in the main text. On the other hand, we assume knowledge of the fundamentals of ordinary differential equations, of measure theory, of functional analysis (distributions, self-adjoint and compact operators in Hilbert spaces, Sobolev-Slobodeckii spaces and their embeddings, interpolation, Nemytskii mapping) and of the calculus of variations (minimizing of coercive, weakly lower semicontinuous functionals). Finally, a section of methodological notes and an index are provided.

We would like to stress that, due to the broadness of the field of superlinear problems, our list of results and methods is of course not complete and is influenced in part by the interests of the authors. For reasons of space, many interesting topics and results could not be mentioned in this book (and we also apologize for any omission.) In particular, we do not touch degenerate problems with superlinear source (involving for instance porous medium, fast diffusion, or p-Laplace operators), nor higher order equations (where the maximum principle does not generally apply). We do not consider superlinear problems involving nonlinear boundary conditions, nor parabolic systems with convection (chemotaxis, Navier-Stokes). These are very interesting and intensively studied topics, but would require a book on their own. Finally, let us mention that there exist several textbooks and monographs dealing, at least in part, with certain aspects of superlinear problems; see [460], [466], [63], [113], [405], [504], [372], [222], for example.

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1. Preliminaries

General

We denote by $B_R(x)$ or $B(x,R)$ the open ball in $\mathbb{R}^n$ with center x and radius R. We set $B_R := B_R(0)$. The $(n-1)$-dimensional unit sphere is denoted by S^{n-1}. The characteristic function of a given set M is denoted by χ_M. We write $D' \subset\subset D$ for $D', D \subset \mathbb{R}^n$ if the closure of D' is a compact subset of D. For any real number s, we set $s_+ := \max(s,0)$ and $s_- := \max(-s,0)$. We also denote $\mathbb{R}_+ := [0,\infty)$.

Domains

Let Ω be a **domain**, i.e. a nonempty, connected, open subset of $\mathbb{R}^n$ and let $k \in \mathbb{N}$. We shall say that Ω is **uniformly regular of class** C^k (cf. [13, p. 642]), if either $\Omega = \mathbb{R}^n$ or there exists a countable family (U_j, φ_j), $j = 1,2,\dots$ of coordinate charts with the following properties:

(i) Each φ_j is a C^k-diffeomorphism of U_j onto the open unit ball B_1 in $\mathbb{R}^n$ mapping $U_j \cap \Omega$ onto the "upper half-ball" $B_1 \cap (\mathbb{R}^{n-1} \times (0,\infty))$ and $U_j \cap \partial\Omega$ onto the flat part $B_1 \cap (\mathbb{R}^{n-1} \times \{0\})$. In addition, the functions φ_j and the derivatives of φ_j and φ_j^{-1} up to the order k are uniformly bounded on U_j and B_1, respectively.

(ii) The set $\bigcup_j \varphi_j^{-1}(B_{1/2})$ contains an ε-neighborhood of $\partial\Omega$ in $\overline{\Omega}$ for some $\varepsilon > 0$.

(iii) There exists $k_0 \in \mathbb{N}$ such that any $k_0 + 1$ distinct sets U_j have an empty intersection.

In an analogous way we define a uniformly regular domain of class $C^{2+\alpha}$ (shortly domain of class $C^{2+\alpha}$). Unless explicitly stated otherwise[1], we will always assume that

$\Omega \subset \mathbb{R}^n$ *is a uniformly regular domain of class* $C^{2+\alpha}$ *for some* $\alpha \in (0,1)$.

On the other hand, we do not assume Ω to be bounded unless this is explicitly mentioned.

We denote the distance to the boundary function by

$$\delta(x) := \operatorname{dist}(x, \partial\Omega).$$

The exterior unit normal on $\partial\Omega$ at a point $x \in \partial\Omega$ is denoted by $\nu(x)$, and the outer normal derivative by ∂_ν or $\partial/\partial\nu$. The surface measure (on e.g. $\partial\Omega$ or S^{n-1}) will be denoted by $d\sigma$ or $d\omega$.

For a given domain Ω and $0 < T < \infty$, we set

$$\begin{aligned} Q_T &:= \Omega \times (0,T), \\ S_T &:= \partial\Omega \times (0,T) \qquad &&\text{(lateral boundary)}, \\ \mathcal{P}_T &:= S_T \cup (\overline{\Omega} \times \{0\}) \qquad &&\text{(parabolic boundary)}. \end{aligned}$$

[1] In fact, if we want to allow nonsmooth domains, we will refer to an *arbitrary* domain.

Functions of space and time

Let $u = u(x,t)$ be a real function of the space variable $x \in \Omega$ and the time variable t. Without fearing confusion we will also consider u as a function of a single variable t with values in a space of functions defined in Ω, hence $u(t)(x) = u(x,t)$.

By a solution of a PDE being **positive** we usually mean that $u(x) > 0$ or $u(x,t) > 0$ in the domain under consideration. Note that, due to the strong maximum principles in Appendix **F**, positive is often equivalent to nontrivial nonnegative.

Radial functions. We say that a domain $\Omega \subset \mathbb{R}^n$ is symmetric if either $\Omega = \mathbb{R}^n$, or $\Omega = B_R = \{x \in \mathbb{R}^n : |x| < R\}$, or $\Omega = \{x \in \mathbb{R}^n : R < |x| < R'\}$, where $0 < R < R' \leq \infty$ (an annulus if $R' < \infty$). Denote $r = |x|$ and let $J \subset \mathbb{R}$ be an interval. A function u defined on a symmetric domain Ω (resp., on $\Omega \times J$) is said to be **radially symmetric**, or simply **radial**, if it can be written in the form $u = u(r)$ (resp., $u = u(r,t)$ for each $t \in J$). The function u is said to be **radial nonincreasing** if it is radial and if, moreover, u is nonincreasing as a function of r.

Banach spaces and linear operators

If X is a Banach space and $p \geq 1$, then X' and p' denote the (topological) dual space and dual exponent $(1/p + 1/p' = 1)$, respectively. We write $X \hookrightarrow Y$ or $X \hookrightarrow\hookrightarrow Y$ if X is continuously or compactly embedded in Y, respectively. If both $X \hookrightarrow Y$ and $Y \hookrightarrow X$ (that is X and Y coincide and carry equivalent norms), then we write $X \doteq Y$. We denote by $\mathcal{L}(X,Y)$ the space of continuous linear operators $A : X \to Y$, $\mathcal{L}(X) = \mathcal{L}(X,X)$. If A is a linear operator in X with the domain of definition $D(A)$ and $Y \subset X$, then the operator A_Y, the **Y-realization of** A, is defined by $A_Y u = Au$, $D(A_Y) := \{u \in D(A) \cap Y : Au \in Y\}$.

Function spaces

We denote by $\mathcal{D}(\Omega)$ the space of C^∞-functions with compact support in Ω. The norms in the Sobolev space $W^{k,p}(\Omega)$ (or the Sobolev-Slobodeckii space $W^{k,p}(\Omega)$ if k is not an integer) and the Lebesgue space $L^p(\Omega)$ will be denoted by $\|\cdot\|_{k,p}$ and $\|\cdot\|_p$, respectively. We denote by $W_0^{1,2}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{1,2}(\Omega)$. The spaces $W^{k,2}(\Omega)$, $k \in \mathbb{N}$, and $W_0^{1,2}(\Omega)$ will also be denoted as $H^k(\Omega)$ and $H_0^1(\Omega)$, respectively. The functions in these spaces are usually understood to be real valued. If no confusion is likely, we shall use the same notation for similar spaces of functions with values in $\mathbb{R}^n$. Otherwise we shall use the notation $L^p(\Omega, \mathbb{R}^n)$, for example.

Let Ω be a bounded domain in $\mathbb{R}^n$ (not necessarily smooth). The weighted Lebesgue spaces $L^p_\delta(\Omega)$ are defined as follows. Denoting as before

$$\delta(x) = \operatorname{dist}(x, \partial\Omega), \qquad x \in \Omega,$$

we put, for all $1 \leq p \leq \infty$,

$$L^p_\delta = L^p_\delta(\Omega) := L^p(\Omega; \delta(x)\,dx).$$

For $1 \leq p < \infty$, L^p_δ is endowed with the norm

$$\|u\|_{p,\delta} = \Big(\int_\Omega |u(x)|^p\,\delta(x)\,dx\Big)^{1/p}.$$

Remark 1.1. Let us note that $L^\infty_\delta(\Omega) = L^\infty(\Omega)$, with same norm. Indeed, $L^\infty_\delta(\Omega)$ consists, by definition, of those measurable functions that are essentially bounded with respect to the measure $\delta(x)\,dx$. $\square$

For any $1 \leq p < \infty$, the uniformly local Lebesgue space (cf. [297], [253]) L^p_{ul} is defined by

$$L^p_{ul} = L^p_{ul}(\mathbb{R}^n) = \big\{\phi \in L^p_{loc}(\mathbb{R}^n) : \|\phi\|_{p,ul} < \infty\big\},$$

where

$$\|\phi\|_{p,ul} := \sup_{a \in \mathbb{R}^n} \Big(\int_{|y-a|<1} |\phi(y)|^p\,dy\Big)^{1/p}.$$

These are Banach spaces with the norm $\|.\|_{p,ul}$. Also, for $p = \infty$, we define $L^\infty_{ul} := L^\infty = L^\infty(\mathbb{R}^n)$. We note that $L^r_{ul} \hookrightarrow L^p_{ul}$ whenever $1 \leq p \leq r \leq \infty$.

In what follows X denotes a Banach space.

Let M be a metric space. Then $B(M,X)$, $BC(M,X)$, $BUC(M,X)$ denote the Banach spaces of bounded, bounded and continuous, bounded and uniformly continuous functions $u : M \to X$, respectively, all endowed with the sup-norm

$$\|u\|_\infty = \|u\|_{\infty,M} := \sup_{t \in M} \|u(t)\|_X.$$

We denote by $C(M,X)$ the space of continuous functions endowed with the topology of locally uniform convergence. If M is locally compact, then we denote by $C_0(M,X)$ the space of functions $u \in BUC(M,X)$ with the following property: Given $\varepsilon > 0$, there exists a compact set $K \subset M$ such that $\|u(t)\|_X < \varepsilon$ for all $t \in M \setminus K$. We also set $B(M) := B(M,\mathbb{R})$, $BC(M) := BC(M,\mathbb{R})$, etc.

Let $M \subset \mathbb{R}^n$. A function $u : M \to X$ is said to be locally Hölder continuous if, for each point $t \in M$, there exist $\alpha \in (0,1)$, $C > 0$ and a neighborhood V of t, such that

$$\lfloor u \rfloor_{\alpha, M\cap V} := \sup_{x,y \in M\cap V,\ x \neq y} \frac{\|u(x) - u(y)\|_X}{|x-y|^\alpha} < \infty. \tag{1.1}$$

If α in (1.1) can be chosen independent of $t \in M$, then u is said to be locally α-Hölder continuous. The space of such functions is denoted by $C^\alpha(M,X)$ (or $C^\alpha(M)$ if $X = \mathbb{R}$) and endowed with the family of seminorms $\|\cdot\|_{\infty,K} + \lfloor\cdot\rfloor_{\alpha,K}$, where K runs over all compact subsets of M. By $UC^\alpha(M,X)$, $\alpha \in (0,1)$, we denote the set of functions $u : M \to X$ such that

$$\lfloor u \rfloor_\alpha := \lfloor u \rfloor_{\alpha,M} < \infty.$$

The norm in the Banach space $BUC^\alpha(M,X) = B(M,X) \cap UC^\alpha(M,X)$ is the sum of the sup-norm and the seminorm $\lfloor\cdot\rfloor_\alpha$. Note that if M is compact, then any locally Hölder continuous function $u : M \to X$ belongs to $BUC^\alpha(M,X)$ for some α and $C^\alpha(M,X) = BUC^\alpha(M,X)$.

If Ω is an arbitrary domain in $\mathbb{R}^n$, then $BC^1(\overline{\Omega})$ denotes the space of functions $u \in BC(\overline{\Omega})$ whose first derivatives in Ω are bounded, continuous and can be continuously extended to $\overline{\Omega}$. The norm of a function u in this space is defined as the sum of sup-norms of u and its first-order derivatives. The spaces $BC^k(\overline{\Omega})$ and $BUC^k(\Omega)$, $k \geq 1$ integer, are defined in an obvious way. If no confusion is likely, we shall denote their norms by $\|\cdot\|_{BC^k}$. The spaces $C^{k+\alpha}(\Omega)$, $UC^{k+\alpha}(\Omega)$, $BUC^{k+\alpha}(\Omega)$, where $k \geq 1$ is an integer and $\alpha \in (0,1)$ are defined similarly.

Let Ω be a bounded domain in $\mathbb{R}^n$. Then $\overline{\Omega}$ is compact, hence any function in $C(\overline{\Omega})$ is bounded and uniformly continuous. On the other hand, the functions in $BUC(\Omega)$ can be uniquely extended to functions in $C(\overline{\Omega})$. Identifying the function $u \in BUC(\Omega)$ with its extension and endowing the space $C(\overline{\Omega})$ with the sup-norm, we can write $BUC(\Omega) = C(\overline{\Omega})$. Similarly, $BUC^\alpha(\Omega) = C^\alpha(\overline{\Omega})$.

If $Q \subset \mathbb{R}^n \times \mathbb{R}$ is a domain in space and time, then $C^{2,1}(Q)$ is the space of functions which are twice continuously differentiable in the spatial variable x and once in the time variable t. The space $BC^{2,1}(\overline{Q})$ has obvious meaning. If $u \in L^p(Q)$, then u_t, $D_x u$ and $D_x^2 u$ denote the time derivative and first and second spatial derivatives of u in the sense of distributions. Alternatively, we shall also use the notation $\nabla u, D^2 u$ instead of $D_x u, D_x^2 u$. We denote by $W^{2,1;p}(Q)$ the space of functions $u \in L^p(Q)$ satisfying $u_t, D_x u, D_x^2 u \in L^p(Q)$, endowed with the norm

$$\|u\|_{2,1;p} = \|u\|_{2,1;p;Q} := \|u\|_{p;Q} + \|D_x u\|_{p;Q} + \|D_x^2 u\|_{p;Q} + \|u_t\|_{p;Q}.$$

Let $Q = Q_T = \Omega \times (0,T)$ where Ω is an arbitrary domain in $\mathbb{R}^n$ and $T > 0$. Given $\alpha \in (0,1]$ set

$$[f]_{\alpha;Q} = \sup\Big\{ \frac{|f(x,t) - f(y,s)|}{|x-y|^\alpha + |t-s|^{\alpha/2}} : x,y \in \Omega,\ t,s \in (0,T),\ (x,t) \neq (y,s) \Big\}.$$

Let k be a nonnegative integer, $\alpha \in (0,1)$ and $a = k + \alpha$. Then we put

$$|f|_{a;Q} = \sum_{|\beta|+2j \leq k} \sup_Q |D_x^\beta D_t^j f| + \sum_{|\beta|+2j=k} [D_x^\beta D_t^j f]_{\alpha;Q}$$

and $BUC^{a,a/2}(Q) := \{ f : |f|_{a;Q} < \infty \}$. The spaces $UC^{a,a/2}(Q)$ and $C^{a,a/2}(Q)$ are defined analogously as in the case of functions defined in R^n. Note that if $p > n+2$, $a < 2 - (n+2)/p$ and Ω is smooth enough (for example, if Ω satisfies a uniform interior cone condition), then

$$W^{2,1;p}(Q) \hookrightarrow BUC^{a,a/2}(Q); \tag{1.2}$$

see [320, Lemmas II.3.3, II.3.4], [399, Theorem 6.9] and the references therein for this statement and more general embedding and trace theorems for anisotropic spaces. Embedding (1.2) can also be derived by using the interpolation embedding in Proposition 51.3 and embeddings for isotropic spaces.

Eigenvalues and eigenfunctions

If Ω is bounded, then we denote by $\lambda_1, \lambda_2, \dots$ the eigenvalues of $-\Delta$ in $W_0^{1,2}(\Omega)$ and by $\varphi_1, \varphi_2, \dots$ the corresponding eigenfunctions. Recall that $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, $\lambda_k \to \infty$ as $k \to \infty$, that

$$\frac{1}{\lambda_1} = \sup\Big\{ \int_\Omega u^2\,dx : u \in W_0^{1,2}(\Omega),\ \int_\Omega |\nabla u|^2\,dx = 1 \Big\}, \tag{1.3}$$

and that we can choose $\varphi_1 > 0$. Unless explicitly stated otherwise, we shall assume that φ_1 is normalized by

$$\int_\Omega \varphi_1\,dx = 1.$$

We shall often use the fact that if Ω is of class C^2, then there exist constants $c_1, c_2 > 0$ such that

$$c_1\delta(x) \leq \varphi_1(x) \leq c_2\delta(x), \qquad x \in \Omega \tag{1.4}$$

(this is a consequence of $u \in C^1(\overline{\Omega})$ and of Hopf's lemma; cf. Proposition 52.1(iii)).

Further frequent notation

We denote by $G(x,y,t) = G_\Omega(x,y,t)$ the Dirichlet heat kernel; $G_t(x) = G(x,t)$ is the Gaussian heat kernel in $\mathbb{R}^n$. The (elliptic) Dirichlet Green kernel is denoted by $K(x,y) = K_\Omega(x,y)$. We implicitly mean by e^{-tA} the Dirichlet heat semigroup in Ω.

The Dirac distribution at point y will be denoted by δ_y.

We shall use the symbols C, C_1, etc. to denote various positive constants. The dependence of these constants will be made precise whenever necessary.

Definitions of various critical exponents ($p_F, p_{BT}, p_{sg}, p_S, p_{JL}, p_L, 2_*, 2^*, q_c$) and other symbols can be found via the List of Symbols.

Chapter I

Model Elliptic Problems

2. Introduction

In Chapter I, we study the problem

$$\left.\begin{aligned} -\Delta u &= f(x,u), \qquad && x\in\Omega, \\ u &= 0, && x\in\partial\Omega, \end{aligned}\right\} \tag{2.1}$$

where $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function (i.e. $f(\cdot,u)$ is measurable for any $u\in\mathbb{R}$ and $f(x,\cdot)$ is continuous for a.e. $x\in\Omega$). Of course, the boundary condition in (2.1) is not present if $\Omega=\mathbb{R}^n$. We will be mainly interested in the model case

$$f(x,u)=|u|^{p-1}u+\lambda u, \qquad \text{where } p>1 \text{ and } \lambda\in\mathbb{R}. \tag{2.2}$$

Denote by p_S the critical Sobolev exponent,

$$p_S := \begin{cases} \infty & \text{if } n\le 2, \\ (n+2)/(n-2) & \text{if } n>2. \end{cases}$$

We shall refer to the cases $p<p_S$, $p=p_S$ or $p>p_S$ as to (Sobolev) subcritical, critical or supercritical, respectively.

3. Classical and weak solutions

Let u be a solution of (2.1) and $\tilde f(x):=f(x,u(x))$. Then u solves the linear problem

$$\left.\begin{aligned} -\Delta u &= \tilde f \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.1}$$

In what follows we define several types of solutions of the linear problem (3.1) (and, consequently, of (2.1)).

Definition 3.1. (i) We call u a **classical solution** of (3.1) if $\tilde f\in C(\Omega)$, $u\in C^2(\Omega)\cap C(\overline{\Omega})$ and u satisfies the equation and the boundary condition in (3.1) pointwise.

(ii) We call $u \in W_0^{1,2}(\Omega)$ a **variational solution** of (3.1) if $\tilde f \in (W_0^{1,2}(\Omega))'$ and

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \tilde f \varphi \, dx \quad \text{for all } \varphi \in W_0^{1,2}(\Omega). \tag{3.2}$$

(iii) Let Ω be bounded, $u \in L^1(\Omega)$. Set

$$\delta(x) := \operatorname{dist}(x, \partial\Omega) \qquad \text{and} \qquad L^1_\delta(\Omega) := L^1(\Omega, \delta(x)dx).$$

We call u an **L^1-solution** of (3.1) if $\tilde f \in L^1(\Omega)$ and

$$\int_\Omega u(-\Delta\varphi)\, dx = \int_\Omega \tilde f \varphi \, dx \quad \text{for all } \varphi \in C^2(\overline\Omega),\ \varphi = 0 \text{ on } \partial\Omega. \tag{3.3}$$

More generally, we call u an **L^1_δ-solution**, or a **very weak solution**, of (3.1) if $\tilde f \in L^1_\delta(\Omega)$ and (3.3) is satisfied. Note that the definition makes sense since $|\varphi| \le C\delta$ hence $\tilde f \varphi \in L^1(\Omega)$. Existence-uniqueness and properties of L^1_δ solutions of the linear problem (3.1) are studied in Appendix C.

(iv) If $\Omega = \mathbb{R}^n$, then $u \in L^1_{loc}(\Omega)$ is called a **distributional solution** of (3.1) if the integral identity in (3.3) is true for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. □

Remarks 3.2. (i) If we assume that $\tilde f$ is a bounded Radon measure in Ω (instead of $\tilde f \in L^1(\Omega)$), then the definition of an L^1-solution still makes sense and we refer to [20] and the references therein for properties of such solutions.

(ii) If $\tilde f \in L^\infty(\Omega)$, then any classical solution of (3.1) satisfies $u \in W^{2,q}(K)$ for any $K \subset\subset \overline\Omega$ and any $q < \infty$. This is a consequence of Remark 47.4(iii). If we further assume that $\tilde f$ is locally Hölder continuous in $\overline\Omega$, then $u \in C^2(\overline\Omega)$.

(iii) Assume Ω bounded. If $\tilde f \in C(\overline\Omega)$, for example, then any classical solution of (3.1) is also a variational solution (this follows from Remark (ii) and integration by parts). If $\tilde f \in L^2(\Omega)$, then any variational solution is an L^1-solution. Some other relations between various types of solutions defined above will be mentioned below (see also Lemma 47.7 in Appendix A). □

In the following sections we shall often use variational methods in order to prove the solvability of (2.1). Therefore, we derive now a sufficient condition on f which guarantees that any variational solution of (2.1) is classical.

If $n \ge 3$ we set $2^* := p_S + 1 = 2n/(n-2)$, $2_* := (2^*)' = 2n/(n+2)$. Assume that the Carathéodory function f satisfies the following growth assumption

$$|f(x,u)| \le \alpha(x) + C_f(|u| + |u|^p), \quad \alpha \in L^{(p+1)'}(\Omega) + L^2(\Omega),\ C_f > 0,\ p \le p_S. \tag{3.4}$$

This growth condition can be significantly weakened if $n \le 2$ but (3.4) will be sufficient for our purposes; cf. (2.2). Denote

$$F(x,u) := \int_0^u f(x,s)\, ds$$

and

$$E(u) := \frac12 \int_\Omega |\nabla u(x)|^2 \, dx - \int_\Omega F(x, u(x))\, dx. \tag{3.5}$$

Since $p \le p_S$ we have $W^{1,2}(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and the embedding is compact provided $p < p_S$ and Ω is bounded. In addition, the energy functional E is C^1 (continuously Fréchet differentiable) in $W^{1,2}(\Omega)$ and

$$E'(u)\varphi = \int_\Omega \nabla u \cdot \nabla\varphi \, dx - \int_\Omega f(\cdot, u)\varphi \, dx$$

for all $u, \varphi \in W^{1,2}(\Omega)$. In particular, each critical point of E in $W_0^{1,2}(\Omega)$ is a variational solution of (2.1).

The following proposition is essentially due to [96]; our proof closely follows the proof of [505, Lemma B.3].

Proposition 3.3. *Assume* (3.4). *If* $n \ge 3$ *assume also* $\alpha \in L^{n/2}(\Omega)$. *Let* u *be a variational solution of* (2.1). *Then* $u \in L^q(\Omega)$ *for all* $q \in [2, \infty)$.

Proof. Since the assertion is obviously true if $n \le 2$ due to $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$, we may assume $n \ge 3$.

Denote $\tilde f(x) := f(x, u(x))$. Then

$$|\tilde f| \le \alpha + C_f(|u| + |u|^p) \le a + b + 2C_f(|u| + |u|^{p_S}),$$

where $a := \alpha\chi_{|u|>1} \in L^{n/2}(\Omega)$, $b := \alpha\chi_{|u|\le 1}$ and α can be written in the form $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \in L^{(p+1)'}(\Omega)$, $\alpha_2 \in L^2(\Omega)$.

Choose $s \ge 0$ such that $u \in L^{2(s+1)}(\Omega)$. We shall prove that $u \in L^{2^*(s+1)}(\Omega)$ so that an obvious bootstrap argument proves the assertion.

Choose $L > 0$ and set

$$\psi := \min(|u|^s, L), \qquad \varphi := u\psi^2, \qquad \Omega_L := \{x \in \Omega : |u|^s \le L\}.$$

In what follows we denote by C, C_1, C_2 various positive constants which may vary from step to step and which may depend on u, s, α, C_f but which are independent of L. We have

$$\nabla(u\psi) = (1 + s\chi_{\Omega_L})(\nabla u)\psi,$$
$$\nabla\varphi = (1 + 2s\chi_{\Omega_L})(\nabla u)\psi^2,$$

and $\varphi \in W_0^{1,2}(\Omega)$. Therefore, we obtain

$$\begin{aligned}\int_\Omega |\nabla u|^2\psi^2\, dx &\le \int_\Omega \nabla u \cdot \nabla\varphi\, dx = \int_\Omega \tilde f \varphi \, dx = \int_\Omega \tilde f u \psi^2 \, dx \\ &\le C\int_\Omega \big[(a+b)|u|\psi^2 + u^2\psi^2 + |u|^{2^*}\psi^2\big]\, dx \\ &\le C\int_\Omega \big[a u^2\psi^2 + b|u| + |u|^{2s+2} + |u|^{2^*}\psi^2\big]\, dx \\ &\le C\Big(1 + \int_\Omega (a + |u|^{2^*-2})u^2\psi^2\, dx\Big),\end{aligned}$$

where we have used

$$\int_\Omega b|u|\,dx \le \int_\Omega \alpha|u|\,dx \le \int_\Omega (|\alpha_1|+|\alpha_2|)|u|\,dx$$
$$\le \|\alpha_1\|_{(p+1)'}\|u\|_{p+1} + \|\alpha_2\|_2\|u\|_2 = C.$$

Consequently, denoting $v := a + |u|^{2^*-2} \in L^{n/2}(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega |\nabla(u\psi)|^2\,dx &\le C\int_\Omega |\nabla u|^2\psi^2\,dx \le C\Big(1+\int_\Omega vu^2\psi^2\,dx\Big)\\
&\le C\Big(1+K\int_{|v|\le K} u^2\psi^2\,dx + \int_{|v|>K} v(u\psi)^2\,dx\Big)\\
&\le C\Big(1+K\int_\Omega |u|^{2s+2}\,dx + \Big(\int_{|v|>K} v^{n/2}\,dx\Big)^{2/n}\Big(\int_\Omega |u\psi|^{2^*}\,dx\Big)^{(n-2)/n}\Big)\\
&\le C_1(1+K) + C_2\varepsilon_K\int_\Omega |\nabla(u\psi)|^2\,dx,\end{aligned}$$

where $\varepsilon_K := \big(\int_{|v|>K} v^{n/2}\,dx\big)^{2/n} \to 0$ as $K\to+\infty$. Choosing K such that $C_2\varepsilon_K < 1/2$ we arrive at

$$\int_{\Omega_L} |\nabla(|u|^{s+1})|^2\,dx = \int_{\Omega_L} |\nabla(u\psi)|^2\,dx \le 2C_1(1+K).$$

Letting $L\to+\infty$ we get $|u|^{s+1}\in W^{1,2}(\Omega)$, hence $u\in L^{2^*(s+1)}(\Omega)$. □

Corollary 3.4. *If f has the form* (2.2) *with $p \le p_S$, then any variational solution u of* (2.1) *is also a classical solution. Moreover, $u\in C^2(\overline{\Omega})$.*

Proof. The assertion is a consequence of standard regularity results for linear elliptic equations. More precisely, for any $2\le q<\infty$, since $\tilde f := f(u)\in L^q(\Omega)$, Theorem 47.3(i) implies the existence of $\tilde u\in W^{2,q}\cap W_0^{1,q}(\Omega)$ such that $-\Delta\tilde u = \tilde f$. Since $u,\tilde u\in H_0^1(\Omega)$, the maximum principle in Proposition 52.3(i) yields $u=\tilde u$. Due to the embedding $W^{2,q}(\Omega)\subset C^1(\overline{\Omega})$ for $q>n$, we deduce that $\tilde f\in C^1(\overline{\Omega})$. Applying now Theorem 47.3(ii), and Proposition 52.3(i) again, we deduce that $u\in C^2(\overline{\Omega})$. □

As for L^1-solutions, we have the following regularity result (we shall see in Remarks 3.6 below that the growth conditions in Propositions 3.3 and 3.5 are optimal).

Proposition 3.5. *Assume Ω bounded. Let the Carathéodory function f satisfy the growth assumption*

$$|f(x,u)| \le C(1+|u|^p), \quad p < p_{sg}, \tag{3.6}$$

where p_{sg} is defined in (3.8). *Let u be an L^1-solution of* (2.1). *Then $u\in C_0\cap W^{2,q}(\Omega)$ for all finite q.*

Proof. It is based on a simple bootstrap argument. Fix $\rho\in(1,n/(n-2)p)$ and put $\tilde f(x) = f(x,u(x))$. Assume that there holds

$$\tilde f \in L^{\rho^i}(\Omega) \tag{3.7}$$

for some $i\ge 0$ (this is true for $i=0$ by assumption). Since

$$\frac{1}{\rho^i} - \frac{1}{p\rho^{i+1}} = \frac{1}{\rho^i}\Big(1-\frac{1}{p\rho}\Big) < \frac{2}{n},$$

by using Proposition 47.5(i), we obtain $u\in L^{p\rho^{i+1}}(\Omega)$, hence $\tilde f\in L^{\rho^{i+1}}(\Omega)$ due to (3.6). By induction, it follows that (3.7) is true for all integers i. In particular $\tilde f\in L^k(\Omega)$ for some $k>n/2$ and we may apply Proposition 47.5(i) once more to deduce that $u\in L^\infty(\Omega)$. The conclusion then follows similarly as in the proof of Corollary 3.4 (using the uniqueness part of Theorem 49.1 instead of Proposition 52.3). □

Remarks 3.6. (i) **Singular solution**. Define the exponent

$$p_{sg} := \begin{cases}\infty & \text{if } n\le 2,\\ n/(n-2) & \text{if } n>2.\end{cases} \tag{3.8}$$

For $p>p_{sg}$ (hence $n\ge 3$), we let

$$U_*(r) := c_p r^{-2/(p-1)}, \quad r>0, \qquad \text{where } c_p^{p-1} := \frac{2}{(p-1)^2}\big((n-2)p-n\big). \tag{3.9}$$

One can easily check that $u_*(x) := U_*(|x|)$ is a positive, radial distributional solution of the equation $-\Delta u = u^p$ in $\mathbb{R}^n$. This singular solution (hence the notation p_{sg}) plays an important role in the study of the parabolic problem (0.2) with $f(u) = |u|^{p-1}u$ (see for example Theorems 20.5, 22.4 and 23.10).

On the other hand, if we set $u(x) := u_*(x) - c_p$ for $0<|x|\le 1$, $\Omega := B_1(0) = \{x\in\mathbb{R}^n : |x|<1\}$, then it is easy to verify that u is an L^1-solution of (2.1) with $f(x,u) = (u+c_p)^p$. Moreover, u is a variational solution of this problem if $p>p_S$. Hence the condition $p\le p_S$ in Proposition 3.3 is necessary.

(ii) Let $n\ge 3$ and let Ω be bounded, $f\in C^1$, $|f(x,u)|\le C(1+|u|^p)$. The example in (i) shows that an L^1-solution need not be classical if $p>p_{sg}$. In fact, it was proved in [44], [394] that the problem

$$\left.\begin{aligned} -\Delta u &= |u|^{p-1}u && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega,\end{aligned}\right\} \tag{3.10}$$

has a positive unbounded radial L^1-solution $u \in C^2(\overline{\Omega}\setminus\{0\})$ provided $p \in [p_{sg}, p_S)$ and $\Omega = B_1(0)$. See also [404] and the references therein for related nonradial results.

(iii) For the case of L^1_δ-solutions, we shall see in Section 11 that the critical exponent is different, namely $(n+1)/(n-1)$. □

Remark 3.7. Classical vs. very weak solutions for the nonlinear eigenvalue problem. Another type of relations between different notions of solutions appears when one considers the nonlinear eigenvalue problem

$$\left.\begin{aligned} -\Delta u &= \lambda f(u), && x \in \Omega, \\ u &= 0, && x \in \partial\Omega. \end{aligned}\right\} \tag{3.11}$$

Here we assume that $f : [0, \infty) \to (0, \infty)$ is a C^1 nondecreasing, convex function, and $\lambda > 0$. Namely, it was proved in [94] (see also [233] for earlier related results) that if there exists a *very weak* solution of (3.11) for some $\lambda_0 > 0$, then there exists a *classical* solution for all $\lambda \in (0, \lambda_0)$. The proof is based on a perturbation argument relying on a variant of Lemma 27.4 below. As a consequence of this and of results from [305], [142], assuming in addition that $\lim_{u\to\infty} f(u)/u = \infty$, there exists $\lambda^* \in (0, \infty)$ such that:

(i) for $0 < \lambda < \lambda^*$, problem (3.11) has a (unique minimal) classical solution u_λ, and the map $\lambda \mapsto u_\lambda$ is increasing;

(ii) for $\lambda = \lambda^*$, problem (3.11) has a very weak solution defined by $u_{\lambda^*} = \lim_{\lambda\uparrow\lambda^*} u_\lambda$;

(iii) for $\lambda > \lambda^*$, problem (3.11) has no very weak solution.

On the other hand, the solution u_{λ^*} may be classical or singular, depending on the nonlinearity. For instance, in the case $f(u) = (u+1)^p$ with $\Omega = B_R$, (3.11) has a classical solution for $\lambda = \lambda^*$ if and only if $p < p_{JL}$, where p_{JL} is defined in (9.3); in the case $f(u) = e^u$, the condition is replaced with $n \le 9$ (see [293], [369]). Illustrations of these facts appear on the bifurcation diagram in Remark 6.10(ii) (see Figure 3). □

4. Isolated singularities

In this section we study the question of isolated singularities of positive classical solutions to the equation $-\Delta u = u^p$. The following result classifies the possible singular behaviors for subcritical or critical p.

Theorem 4.1. *Let $n \ge 3$ and $1 < p \le p_S$. Assume that u is a positive classical solution of*

$$-\Delta u = u^p \quad in\ B_1 \setminus \{0\} \tag{4.1}$$

and that u is unbounded at 0. Then there exist constants $C_2 \ge C_1 > 0$ such that

$$C_1\psi(x) \le u(x) \le C_2\psi(x), \quad 0 < |x| < 1/2,$$

where

$$\psi(x) = \begin{cases} |x|^{2-n} & \text{if } 1 < p < p_{sg}, \\ |x|^{2-n}(-\log|x|)^{(2-n)/2} & \text{if } p = p_{sg}, \\ |x|^{-2/(p-1)} & \text{if } p_{sg} < p \le p_S. \end{cases}$$

Moreover, if $p < p_S$, then we have $C_2 \le \tilde{C}_2$ with $\tilde{C}_2 = \tilde{C}_2(n,p) > 0$.

Furthermore, for all $p > 1$, we have the following result, which explains in what sense the equation can be extended to the whole unit ball.

Theorem 4.2. *Let $p > 1$ and $n \ge 3$. Assume that u is a positive classical solution of*

$$-\Delta u = u^p \quad in\ B_1 \setminus \{0\}.$$

(i) *Then $u^p \in L^1_{loc}(B_1)$ and there exists $a \ge 0$ such that u is a solution of*

$$-\Delta u = u^p + a\delta_0 \quad in\ \mathcal{D}'(B_1),$$

where δ_0 denotes the Dirac delta distribution. Moreover, we have $a \le \tilde{a}$ with $\tilde{a} = \tilde{a}(n,p) > 0$.

(ii) *If $p < p_{sg}$ and $a = 0$, then the singularity is removable, i.e. u is bounded near $x = 0$.*

(iii) *If $p \ge p_{sg}$, then $a = 0$.*

Remarks 4.3. (i) Theorem 4.1 follows from [340], [44], [240] (see also [83]), and [108], for the cases $p < p_{sg}$, $p = p_{sg}$, $p_{sg} < p < p_S$ and $p = p_S$ respectively. Theorem 4.2 follows from [97] and [340]. See also the book [523] for further results and references.

(ii) Under the assumptions of Theorem 4.1 with $p_{sg} < p < p_S$, it can be shown more precisely that $|x|^{2/(p-1)}u(x) \to c_p$, as $x \to 0$, where c_p is given by (3.9) (cf. [240], [83]). If $1 < p < p_{sg}$, then actually $|x|^{n-2}u(x) \to C > 0$, as $x \to 0$ (see the proof below). Examples in [340] show that singular solutions do exist for $1 < p < p_{sg}$ and that the constant C may depend on the solution u.

(iii) If $p > p_S$, then the upper estimate $u(x) \le C|x|^{-2/(p-1)}$ is still true in the radial case (cf. [213], [397]). In fact, as a consequence of $-(r^{n-1}u_r)_r = r^{n-1}u^p > 0$, for $r > 0$ small, we have either $u_r > 0$, hence u bounded, or $u_r \le 0$. In this second case, by integration, we get $-u_r \ge r^{1-n}\int_0^r s^{n-1}u^p(s)\,ds \ge (r/n)u^p$ for $r > 0$ small, hence $(u^{1-p})_r \ge Cr$, and the upper estimate follows by a further integration. The estimate is unknown in the nonradial case for $p > p_S$, but related integral estimates of solutions can be found in e.g. [238] and [89].

(iv) A result similar to Theorem 4.1 is true for $n = 2$, with $\psi(x)$ given by the fundamental solution $\log|x|$ instead of $|x|^{2-n}$. These results are related to the fact that the (H^1–) capacity of a point is 0 when $n \geq 2$. When the origin is replaced by a closed subset of 0 capacity, related results can be found in [170]. On the other hand, the upper estimate $u(x) \leq C|x|^{-2/(p-1)}$, from the case $p_{sg} < p < p_S$, can be generalized to sets other than a single point (see Theorem 8.7 in Section 8). □

We shall first prove Theorem 4.2. Theorem 4.1 in the case $1 < p < p_{sg}$ will then follow as a consequence of Theorem 4.2 and of a bootstrap argument. In the case $p_{sg} < p < p_S$, the upper estimate will be a consequence of the more general result Theorem 8.7 in Section 8. For the cases $p = p_{sg}$, $p = p_S$, and for the lower estimate when $p_{sg} < p < p_S$, see the above mentioned references.

In view of the proofs, we introduce the following notation. We denote by $\Gamma(x) = c_n|x|^{2-n}$ the fundamental solution of the Laplacian (Newton potential), i.e. $-\Delta\Gamma = \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$. We let $\omega = \{x \in \mathbb{R}^n : |x| < 1/2\}$ and fix $\chi \in \mathcal{D}(B_1)$ such that $\chi = 1$ on ω and $0 \leq \chi \leq 1$. For each positive integer j, denote $\chi_j(x) = \chi(jx)$. By a straightforward calculation using $n \geq 3$, we see that $\chi_j \to 0$ in $H^1(B_1)$ as $j \to \infty$. For any $\varphi \in \mathcal{D}(B_1)$, we put $\varphi_j := (1-\chi_j)\varphi$. Observe that $\varphi_j \to \varphi$ in $H^1(B_1)$.

We need the following lemma.

Lemma 4.4. *Let $n \geq 3$. Assume that $u \in C^2(B_1 \setminus \{0\})$ satisfies $u \geq 0$ and*

$$-\Delta u \geq 0 \quad \text{in } B_1 \setminus \{0\}.$$

Then $u \in L^1_{loc}(B_1)$ and

$$-\Delta u \geq 0 \quad \text{in } \mathcal{D}'(B_1).$$

Proof. For each $k > 0$, we take a function $G_k \in C^2([0,\infty))$ such that $G_k(s) = s$ for $0 \leq s \leq k$, $G_k(s) = k+1$ for s large, $G_k' \geq 0$ and $G_k'' \leq 0$. Define $u_k := G_k(u)$ and note that the sequence $\{u_k\}_k$ is monotone increasing and converges to u pointwise in $B_1 \setminus \{0\}$. The function u_k satisfies

$$-\Delta u_k = -G_k'(u)\Delta u - G_k''(u)|\nabla u|^2 \geq 0 \quad \text{in } B_1 \setminus \{0\}. \tag{4.2}$$

Fix $\alpha > 0$ and $\varphi \in \mathcal{D}(B_1)$. Multiplying inequality (4.2) by the test-function $\varphi_j^2(1+u_k)^{-\alpha}$ and integrating by parts, we obtain

$$\begin{aligned} 0 &\leq \int_{B_1} \nabla u_k \cdot \nabla(\varphi_j^2(1+u_k)^{-\alpha}) \\ &= -\alpha \int_{B_1} |\nabla u_k|^2 \varphi_j^2 (1+u_k)^{-1-\alpha} + 2\int_{B_1} \nabla u_k \cdot \nabla\varphi_j (1+u_k)^{-\alpha}\varphi_j. \end{aligned}$$

It follows that

$$\begin{aligned} &\alpha \int_{B_1} |\nabla u_k|^2 \varphi_j^2 (1+u_k)^{-1-\alpha} \\ &\qquad \leq \frac{\alpha}{2}\int_{B_1} |\nabla u_k|^2 \varphi_j^2 (1+u_k)^{-1-\alpha} + C(\alpha)\int_{B_1} |\nabla\varphi_j|^2 (1+u_k)^{1-\alpha}, \end{aligned}$$

hence

$$\int_{B_1} |\nabla u_k|^2 \varphi_j^2 (1+u_k)^{-1-\alpha} \leq C(\alpha)\int_{B_1} |\nabla\varphi_j|^2 (1+u_k)^{1-\alpha}.$$

Since $|\nabla\varphi_j|^2 \to |\nabla\varphi|^2$ in $L^1(B_1)$ and $(1+u_k)^{1-\alpha} \in L^\infty(B_1)$, we may pass to the limit $j \to \infty$ (using Fatou's lemma on the LHS) and we obtain

$$\int_{B_1} |\nabla u_k|^2 \varphi^2 (1+u_k)^{-1-\alpha} \leq C(\alpha)\int_{B_1} |\nabla\varphi|^2 (1+u_k)^{1-\alpha}.$$

First taking $\alpha = 1$ and using $1 + u_k \leq k+2$, we deduce that $u_k \in H^1(\omega)$, hence $u_k \in H^1_{loc}(B_1)$.

Next take $\alpha = 2/n$. Consider φ such that $\varphi = 1$ for $|x| \leq 1/4$ and with support in ω. Applying the Sobolev and Hölder inequalities, we get, for any $\rho \in (0, 1/2)$,

$$\begin{aligned} \Big(\int_{|x|<1/4} (1+u_k)\Big)^{\frac{n-2}{n}} &\leq C\int_{|x|<1/4} \big|\nabla\big[(1+u_k)^{\frac{n-2}{2n}}\big]\big|^2 + C\int_{|x|<1/4} (1+u_k)^{\frac{n-2}{n}} \\ &\leq C\int_\omega (1+u_k)^{\frac{n-2}{n}} \\ &\leq C\int_{\rho<|x|<1/2} (1+u_k)^{\frac{n-2}{n}} + C\rho^2\Big(\int_{|x|\leq\rho} (1+u_k)\Big)^{\frac{n-2}{n}}. \end{aligned}$$

Since u is bounded on $\{\rho < |x| < 1/2\}$ and $u_k \leq u$, by taking $\rho \in (0, 1/4)$ small enough, we deduce that $\int_{|x|<1/4} u_k \leq C$ independent of k. Consequently $u \in L^1(\omega)$, hence $u \in L^1_{loc}(B_1)$, and $u_k \to u$ in $L^1_{loc}(B_1)$.

Now assuming $\varphi \geq 0$, we multiply inequality (4.2) by φ_j and integrate by parts. We obtain

$$\int_{B_1} \nabla u_k \cdot \nabla\varphi_j = \int_{B_1} (-\Delta u_k)\varphi_j \geq 0.$$

Since $u_k \in H^1_{loc}(B_1)$, we may pass to the limit $j \to \infty$ to get $\int_{B_1} \nabla u_k \cdot \nabla\varphi \geq 0$, hence $\int_{B_1} (-\Delta\varphi)u_k \geq 0$. Since $u_k \to u$ in $L^1_{loc}(B_1)$, we conclude that $\int_{B_1} (-\Delta\varphi)u \geq 0$ and the proof of the lemma is complete. □

Proof of Theorem 4.2. (i) By Lemma 4.4, we know that $u \in L^1_{loc}(B_1)$ and that $-\Delta u \geq 0$ in $\mathcal{D}'(B_1)$. It follows that Δu is a Radon measure (in other words, a 0-order distribution) on ω. Indeed, for each $\varphi \in \mathcal{D}(B_1)$ with $\mathrm{supp}(\varphi) \subset \omega$, using $\|\varphi\|_\infty\chi \pm \varphi \geq 0$, we obtain

$$\langle -\Delta u, \|\varphi\|_\infty\chi \pm \varphi\rangle \geq 0$$

hence

$$|\langle -\Delta u, \varphi\rangle| \leq |\langle -\Delta u, \chi\rangle|\, \|\varphi\|_\infty =: C\|\varphi\|_\infty. \tag{4.3}$$

We next claim that $u^p \in L^1_{loc}(B_1)$. To this end, we assume $\varphi \geq 0$, we multiply (4.1) by φ_j and we integrate by parts. We obtain

$$\int_{B_1} u^p \varphi_j = \langle -\Delta u, \varphi_j \rangle \leq C\|\varphi_j\|_\infty \leq C\|\varphi\|_\infty,$$

due to (4.3), and the claim follows from Fatou's lemma.

Now a classical argument in distribution theory allows us to conclude: Denote $T = \Delta u + u^p \in \mathcal{D}'(B_1)$ and let $\varphi \in \mathcal{D}(B_1)$. Since $T = 0$ in $\mathcal{D}'(B_1 \setminus \{0\})$ and $(1-\chi_j)\varphi = 0$ in the neighborhood of 0, we have $\langle T, (1-\chi_j)\varphi \rangle = 0$. Consequently,

$$\langle T, \varphi \rangle - \langle T, \chi_j \rangle \varphi(0) = \langle T, \varphi\chi_j \rangle - \langle T, \chi_j \rangle \varphi(0) = \langle T, (\varphi - \varphi(0))\chi_j \rangle. \tag{4.4}$$

But since $\|(\varphi - \varphi(0))\chi_j\|_\infty \to 0$ as $j \to \infty$, it follows that the LHS of (4.4) converges to 0 as $j \to \infty$. We first deduce that $\ell = \lim_{j\to\infty} \langle T, \chi_j \rangle$ exists (take a φ such that $\varphi(0) \neq 0$). Moreover, since $-\Delta u \geq 0$ in $\mathcal{D}'(B_1)$, we have $\ell \leq \lim_{j\to\infty} \int_\omega u^p \chi_j = 0$ by dominated convergence. Returning to (4.4), we obtain

$$\Delta u + u^p = -a\delta_0 \tag{4.5}$$

with $a = -\ell \geq 0$. Now let $\psi \in \mathcal{D}(B_1)$ satisfy $-\Delta\psi \leq \mu\psi^{1/p}$ in B_1 for some $\mu > 0$ and $\psi \geq C > 0$ for $|x| < 2/3$ (such function is given for instance by $\psi(x) = \exp[-(1-2|x|^2)^{-1}]$ for $|x|^2 < 1/2$ and $\psi(x) = 0$ otherwise). Testing equation (4.5) with ψ, we get

$$a\psi(0) + \int_{B_1} u^p \psi = -\int_{B_1} u\Delta\psi \leq \mu \int_{B_1} u\psi^{1/p} \leq \frac{1}{2}\int_{B_1} u^p\psi + C(p,n).$$

It follows that

$$a + \int_{\{|x|<2/3\}} u^p < \tilde{a}(n,p). \tag{4.6}$$

In particular, assertion (i) is proved.

For further reference, we also observe that

$$u \geq a\Gamma - C \quad \text{in } \omega. \tag{4.7}$$

To show this, we first note that $v := u - a\Gamma$ satisfies $-\Delta v = u^p$ in $\mathcal{D}'(B_1)$. By Lemma 47.7, $w := \chi v$ is an L^1-solution of

$$\left.\begin{aligned} -\Delta w &= g := u^p\chi - h && \text{in } B_1,\\ w &= 0 && \text{on } \partial B_1, \end{aligned}\right\} \tag{4.8}$$

where $h := 2\nabla u \cdot \nabla\chi + u\Delta\chi \in L^\infty(B_1)$. At this point, let us introduce the function $\Theta \in C^2(\overline{B}_1)$, $\Theta \geq 0$, classical solution of the problem

$$\left.\begin{aligned} -\Delta\Theta &= 1 && \text{in } B_1,\\ \Theta &= 0 && \text{on } \partial B_1. \end{aligned}\right\}$$

(This is the so-called "torsion" function, which will be useful as a comparison or test-function later again.) Then $w + \|h\|_\infty\Theta$ is an L^1-solution of (4.8) with g replaced by $g + \|h\|_\infty \geq 0$. By the maximum principle part of Theorem 49.1, we deduce that $w + \|h\|_\infty\Theta \geq 0$, hence (4.7).

(ii) Let $1 < p < p_{sg}$ and assume that $a = 0$. We have seen that $w = \chi u$ is an L^1-solution of (4.8). Moreover, since $\chi = 1$ near $x = 0$, we may write $g = w^p + \tilde{h}$ in (4.8) for some $\tilde{h} \in L^\infty(B_1)$. It then follows from Proposition 3.5 that $w \in L^\infty(B_1)$, hence $u \in L^\infty(\omega)$.

(iii) Assume $p \geq p_{sg}$. If we had $a > 0$, then (4.7) would imply $u^p \geq C|x|^{-(n-2)p}$ as $x \to 0$ for some $C > 0$. Since $u^p \in L^1_{loc}(B_1)$ due to (i), we conclude that $a = 0$. □

Proof of Theorem 4.1 for $1 < p < p_{sg}$. By Theorem 4.2, we know that

$$-\Delta u = u^p + a\delta_0 \quad \text{in } \mathcal{D}'(B_1)$$

with $a > 0$. Denote $v_0 = u$, $\alpha_1 = n-2$, and put $v_1 := u - a\Gamma = v_0 - C_1|x|^{-\alpha_1}$. Then we have

$$-\Delta v_1 = u^p \quad \text{in } \mathcal{D}'(B_1).$$

On the other hand, an easy calculation shows that $-\Delta(|x|^{-\alpha}) = C(\alpha)|x|^{-\alpha-2}$ in $\mathcal{D}'(B_1)$ for all $\alpha \in (0, n-2)$ and some $C(\alpha) > 0$. Set $\alpha_2 := p\alpha_1 - 2$ if $p\alpha_1 > 2$ and choose $\alpha_2 \in (0, \alpha_1)$ otherwise. Notice that $\alpha_2 \in (0, \alpha_1) = (0, n-2)$ in both cases due to $p\alpha_1 < n$. Since $u^p \leq C(v_1)_+^p + C|x|^{-p\alpha_1} \leq C(v_1)_+^p + C|x|^{-\alpha_2-2}$, there exists $C_2 > 0$ such that $v_2 := v_1 - C_2|x|^{-\alpha_2}$ satisfies

$$-\Delta v_2 \leq C(v_1)_+^p \quad \text{in } \mathcal{D}'(B_1).$$

Since $(v_1)_+^p \leq C(v_2)_+^p + C|x|^{-p\alpha_2}$, we can iterate this procedure and we obtain functions v_i $(i = 0, 1, \dots)$ satisfying $v_{i+1} = v_i - C_{i+1}|x|^{-\alpha_{i+1}}$, with $\alpha_{i+1} \in (0, \alpha_i)$, and

$$-\Delta v_{i+1} \leq C_i'(v_i)_+^p \quad \text{in } \mathcal{D}'(B_1).$$

Moreover, due to $0 < a \leq \tilde{a}(n,p)$, the constants C_i, C_i' may be chosen to depend only on n, p, i.

To conclude, we apply a bootstrap argument similar to that in the proof of Proposition 3.5: Fix $\rho \in (1, n/(n-2)p)$, let $\Omega_1 = \{|x| < 2/3\}$, and assume that $(v_i)_+ \in L^{p\rho^i}_{loc}(\Omega_1)$ for some $i \geq 0$ (this is true for $i = 0$ in view of (4.6)). Since $(-\Delta v_{i+1})_+ \in L^{\rho^i}_{loc}(\Omega_1)$ and

$$\frac{1}{\rho^i} - \frac{1}{p\rho^{i+1}} = \frac{1}{\rho^i}\Big(1 - \frac{1}{p\rho}\Big) < \frac{2}{n},$$

we may apply Proposition 47.6(ii) to deduce that $(v_{i+1})_+ \in L^{p\rho^{i+1}}_{loc}(\Omega_1)$. By iterating, we get $(v_i)_+ \in L^k_{loc}(\Omega_1)$ for some sufficiently large i and some $k > n/2$. We may then apply Proposition 47.6(ii) once more to deduce that $(v_{i+1})_+ \in L^\infty(\omega)$. This implies

$$u - a\Gamma = v_1 = v_{i+1} + \sum_{j=2}^{i+1} C_j|x|^{-\alpha_j} \le C(1+|x|^{-\alpha_2}), \quad |x| < 1/2.$$

Moreover, starting from (4.6), it is easy to check that the constant C depends only on n, p. This along with (4.7) yields the conclusion. □

5. Pohozaev's identity and nonexistence results

In this section we prove the nonexistence of nontrivial solutions of (2.1) provided f satisfies (2.2) with $p \ge p_S$, $\lambda \le 0$ and Ω is a bounded starshaped domain. The following identity [419] plays a crucial role in the proof.

Theorem 5.1. *Let u be a classical solution of* (2.1) *with $f = f(u)$ being locally Lipschitz and Ω bounded. Then*

$$\frac{n-2}{2}\int_\Omega |\nabla u|^2\,dx - n\int_\Omega F(u)\,dx + \frac12\int_{\partial\Omega}\Big|\frac{\partial u}{\partial\nu}\Big|^2 x\cdot\nu\,d\sigma = 0, \tag{5.1}$$

where $F(u) = \int_0^u f(s)ds$.

Proof. First notice that $u \in C^2(\overline{\Omega})$ (see Remark 3.2(ii)). Using integration by parts we obtain

$$\begin{aligned}
&\int_\Omega \big[\nabla u\cdot\nabla(x\cdot\nabla u) - |\nabla u|^2\big]\,dx \\
&\quad = \int_\Omega \sum_{i,j}\frac{\partial u}{\partial x_i}x_j\frac{\partial}{\partial x_i}\Big(\frac{\partial u}{\partial x_j}\Big)\,dx = \int_\Omega \sum_{i,j}\frac{\partial u}{\partial x_i}x_j\frac{\partial}{\partial x_j}\Big(\frac{\partial u}{\partial x_i}\Big)\,dx \\
&\quad = \int_{\partial\Omega}\sum_{i,j}\frac{\partial u}{\partial x_i}x_j\frac{\partial u}{\partial x_i}\nu_j\,d\sigma - n\int_\Omega |\nabla u|^2\,dx - \int_\Omega\sum_{i,j}\frac{\partial}{\partial x_j}\Big(\frac{\partial u}{\partial x_i}\Big)x_j\frac{\partial u}{\partial x_i}\,dx,
\end{aligned}$$

hence

$$\int_\Omega \sum_{i,j}\frac{\partial u}{\partial x_i}x_j\frac{\partial}{\partial x_j}\Big(\frac{\partial u}{\partial x_i}\Big)\,dx = \frac12\Big(\int_{\partial\Omega}\Big|\frac{\partial u}{\partial\nu}\Big|^2 x\cdot\nu\,d\sigma - n\int_\Omega|\nabla u|^2\,dx\Big)$$

and

$$\int_\Omega \nabla u\cdot\nabla(x\cdot\nabla u)\,dx = \frac12\int_{\partial\Omega}\Big|\frac{\partial u}{\partial\nu}\Big|^2 x\cdot\nu\,d\sigma - \frac{n-2}{2}\int_\Omega|\nabla u|^2\,dx.$$

Multiplying the equation in (2.1) by $x\cdot\nabla u$, integrating over Ω and denoting the left- or the right-hand side of the resulting equation by (LHS) or (RHS), respectively, we obtain

$$\begin{aligned}
-(\mathrm{LHS}) &= \int_\Omega \Delta u(x\cdot\nabla u)\,dx = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}(x\cdot\nabla u)\,d\sigma - \int_\Omega \nabla u\cdot\nabla(x\cdot\nabla u)\,dx \\
&= \frac12\int_{\partial\Omega}\Big|\frac{\partial u}{\partial\nu}\Big|^2 x\cdot\nu\,d\sigma + \frac{n-2}{2}\int_\Omega|\nabla u|^2\,dx, \\
(\mathrm{RHS}) &= \int_\Omega f(u)(x\cdot\nabla u)\,dx \\
&= \int_{\partial\Omega}F(u)(x\cdot\nu)\,d\sigma - n\int_\Omega F(u)\,dx = -n\int_\Omega F(u)\,dx,
\end{aligned}$$

where we have used that, on $\partial\Omega$, $\nabla u = C\nu$ for suitable $C \in \mathbb{R}$ and $F(u) = F(0) = 0$. The comparison of (LHS) and (RHS) yields now the assertion. □

Corollary 5.2. *Assume Ω bounded and starshaped with respect to some point $x_0 \in \Omega$ (i.e. the segment $[x_0, x]$ is a subset of Ω for any $x \in \Omega$), $n \ge 3$. Assume that*

$$F(u) \le \frac{n-2}{2n}f(u)u \qquad \text{for all } u. \tag{5.2}$$

Then (2.1) *does not possess classical positive solutions. If, in addition, $f(0) = 0$, then* (2.1) *does not possess classical nontrivial solutions.*

Condition (5.2) *is satisfied if, for example, $f(u) = |u|^{p-1}u + \lambda u$, $p \ge p_S$ and $\lambda \le 0$.*

Proof. We proceed by contradiction. We can assume that Ω is starshaped with respect to $x_0 = 0$. Then $x\cdot\nu \ge 0$ on $\partial\Omega$ and

$$\int_{\partial\Omega} x\cdot\nu\,d\sigma = \int_\Omega \Delta\Big(\frac{x^2}{2}\Big)\,dx > 0,$$

hence $x\cdot\nu > 0$ on a set of positive surface measure in $\partial\Omega$.

If u is a positive solution of (2.1), then $\partial u/\partial\nu < 0$ on $\partial\Omega$ by the maximum principle and we obtain

$$\frac12\int_{\partial\Omega}\Big|\frac{\partial u}{\partial\nu}\Big|^2 x\cdot\nu\,d\sigma > 0. \tag{5.3}$$

Multiplication of the equation in (2.1) by u and integration by parts yields

$$\int_\Omega |\nabla u|^2\,dx = \int_\Omega f(u)u\,dx \tag{5.4}.$$

Using (5.1), (5.3), (5.4) we arrive at

$$\int_\Omega \Big[\frac{n-2}{2n}f(u)u - F(u)\Big]\,dx < 0,$$

which contradicts (5.2).

If $f(0) = 0$ and u is a sign-changing solution of (2.1), then the assertion follows from the unique continuation property. In fact, let $x_1 \in \partial\Omega$ be such that $x \cdot \nu > 0$ in a neighborhood Γ_1 of x_1 in $\partial\Omega$ (recall that $\partial\Omega$ is smooth). Then the above arguments guarantee $\partial u/\partial\nu = 0$ on Γ_1. Since $u = 0$ and $\Delta u = f(u) = 0$ on $\partial\Omega$, all the second derivatives of u have to vanish on Γ_1. Set $u(x) := 0$ for $x \notin \overline{\Omega}$. Then u is a solution of (2.1) in a neighborhood of Γ_1, hence $u \equiv 0$ in this neighborhood due to [272, Satz 2]. Using the same result one can easily show $u \equiv 0$ in Ω. □

Remark 5.3. The idea of considering the multiplier $x \cdot \nabla u$ was used before in [455] in the linear case $f(u) = \mu u$ (for a different purpose, namely an integral representation of the eigenvalues of the Laplacian). Identities similar to (5.1) (see Lemma 31.4 for the case of systems and see also [431]) are sometimes called Rellich-Pohozaev type identities in the literature. □

6. Homogeneous nonlinearities

In this section we use variational methods in order to study the problem

$$\left.\begin{aligned} -\Delta u &= |u|^{p-1}u + \lambda u, && x \in \Omega, \\ u &= 0, && x \in \partial\Omega. \end{aligned}\right\} \tag{6.1}$$

The energy functional E has the form $E(u) = \Psi(u) - \Phi(u)$, where

$$\Psi(u) := \frac{1}{2}\int_\Omega \big[|\nabla u(x)|^2 - \lambda u^2\big]\,dx \quad \text{and} \quad \Phi(u) := \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx. \tag{6.2}$$

Notice that Ψ is quadratic and Φ is positively homogeneous of order $p+1 \neq 2$. Therefore, if

$$\Psi'(w) = \mu\Phi'(w) \tag{6.3}$$

for some $\mu > 0$, then, setting $t := \mu^{1/(p-1)}$, we get

$$E'(tw) = \Psi'(tw) - \Phi'(tw) = t\big[\Psi'(w) - t^{p-1}\Phi'(w)\big] = 0. \tag{6.4}$$

Consequently, tw is a critical point of E, hence a classical solution of (6.1) if $p \le p_S$. A nontrivial function w satisfying (6.3) will be found by minimizing the functional Ψ with respect to the set $M := \{u : \Phi(u) = 1\}$ and using the following well-known Lagrange multiplier rule.

Theorem 6.1. *Let X be a real Banach space, $w \in X$ and let $\Psi, \Phi_1, \dots, \Phi_k : X \to \mathbb{R}$ be C^1 in a neighborhood of w. Denote $M := \{u \in X : \Phi_i(u) = \Phi_i(w)$ for $i = 1, \dots, k\}$ and assume that w is a local minimizer of Ψ with respect to the set M. If $\Phi_1'(w), \dots, \Phi_k'(w)$ are linearly independent, then there exist $\mu_1, \dots, \mu_k \in \mathbb{R}$ such that*

$$\Psi'(w) = \sum_{i=1}^k \mu_i \Phi_i'(w).$$

Our proofs of the main results of this section (Theorem 6.2 and Theorem 6.7(i)) follow those in [505, Theorem I.2.1 and Lemma III.2.2]. Let us first consider the subcritical case.

Theorem 6.2. *Assume Ω bounded. Let $1 < p < p_S$ and $\lambda < \lambda_1$. Then there exists a positive classical solution of* (6.1).

Proof. Set $X := W_0^{1,2}(\Omega)$ and define Ψ, Φ as in (6.2). Since

$$\Psi''(u)[h,h] = 2\Psi(h) \ge c_\lambda \int_\Omega |\nabla h|^2\,dx, \quad c_\lambda := 1 - \frac{\lambda}{\lambda_1} > 0,$$

the functional Ψ is convex and coercive. Let $u_k \in M := \{u \in X : \Phi(u) = 1\}$, $u_k \rightharpoonup u$ in X. Then $u_k \to u$ in $L^{p+1}(\Omega)$ due to $X \hookrightarrow\hookrightarrow L^{p+1}(\Omega)$, hence $u \in M$. Consequently, the set M is weakly sequentially closed in the reflexive space X and there exists $w \in M$ such that $\Psi(w) = \inf_M \Psi$. Since $|w| \in M$ and $\Psi(|w|) = \Psi(w)$, we may assume $w \ge 0$. Moreover, $\Phi'(w)w = (p+1)\Phi(w) = p+1$, hence $\Phi'(w) \neq 0$. Theorem 6.1 guarantees the existence of $\mu \in \mathbb{R}$ such that $\Psi'(w) = \mu\Phi'(w)$, hence

$$0 < 2\Psi(w) = \Psi'(w)w = \mu\Phi'(w)w = \mu(p+1)\Phi(w) = \mu(p+1).$$

Consequently, $\mu > 0$ and we deduce from (6.4) that $u := \mu^{1/(p-1)}w$ is a nonnegative variational solution of (6.1), $u \not\equiv 0$. Corollary 3.4 guarantees that u is a classical solution and the strong maximum principle shows $u > 0$ in Ω. □

Remarks 6.3. (i) **Annulus.** Assume that $\Omega = \{x \in \mathbb{R}^n : 1 < |x| < 2\}$, $\lambda < \lambda_1$ and let X denote the space all of radial functions in $W_0^{1,2}(\Omega)$. It is easily seen that X is compactly embedded into the space Y of all radial functions in $L^{p+1}(\Omega)$ for any $p > 1$ (in fact, X and Y are isomorphic to $W_0^{1,2}\big((1,2)\big)$ and $L^{p+1}\big((1,2)\big)$, respectively). Moreover, any critical point of E in X is obviously a classical solution of (6.1). Hence the proof of Theorem 6.2 guarantees the existence of a positive classical solution of (6.1) for all $p > 1$.

(ii) **Nonexistence for $\lambda \ge \lambda_1$.** If Ω is bounded, $\lambda \ge \lambda_1$ and $p > 1$ is arbitrary, then (6.1) does not have positive stationary solutions. To see this, it is sufficient to multiply the equation in (6.1) by the first eigenfunction φ_1 to obtain

$$0 = \int_\Omega |u|^{p-1}u\varphi_1\,dx + (\lambda - \lambda_1)\int_\Omega u\varphi_1\,dx > 0$$

provided u is a positive solution. □

Remark 6.4. Unbounded domains. Let $\Omega = \mathbb{R}^n$, $1 < p < p_S$ and $\lambda < 0$ (notice that 0 is the minimum of the spectrum of $-\Delta$ in $W^{1,2}(\mathbb{R}^n)$). Let X and Y denote the space of radial functions in $W^{1,2}(\mathbb{R}^n)$ and $L^{p+1}(\mathbb{R}^n)$, respectively. If $n \geq 2$, then X is compactly embedded in Y (see [76, Theorem A.I']) so that we may use the approach above in order to get a positive solution of (6.1). Moreover, using Schwarz symmetrization it is easy to see that the minimizer of $\Psi(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda u^2\right) dx$ in $M_X := \{u \in X : \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx = 1\}$ is also a minimizer in the larger set $M := \{u \in W^{1,2}(\mathbb{R}^n) : \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx = 1\}$.

In the case $\Omega = \mathbb{R}^n$ one can use a similar approach to that used in Theorem 6.2 for functions $f = f(u)$ (or $f = f(|x|, u)$) which need not be homogeneous. In fact, if one is able to find a minimizer u of $\frac{1}{2}\int_\Omega |\nabla u|^2\,dx$ in the set $N := \{u \in X : \int_\Omega F(u)\,dx = 1\}$, then there exists $\sigma > 0$ such that the function $u_\sigma(x) := u(x/\sigma)$ solves (6.1). This idea was used in [76], for example. For more recent results on existence and uniqueness of positive solutions of this problem with $f = f(u)$ we refer to [235], [420] and the references therein.

If f depends on x (and not only on $|x|$) or if Ω is unbounded and not symmetric, then the situation is more delicate. In some cases, one can use the concentration compactness arguments in order to get a solution (see [50] and the references therein). □

Let us now turn to the critical case $p = p_S$. In view of Corollary 5.2 and the proof of Theorem 6.2, the functional Ψ cannot attain its infimum over the set M if Ω is starshaped and $\lambda = 0$. In other words, denoting

$$\begin{aligned} S_\lambda(u,\Omega) &:= \frac{\int_\Omega \left[|\nabla u|^2 - \lambda|u|^2\right] dx}{\|u\|_{2^*}^2}, \\ S_\lambda(\Omega) &:= \inf\{S_\lambda(u,\Omega) : u \in W_0^{1,2}(\Omega),\ u \neq 0\} \\ &= \inf\{S_\lambda(u,\Omega) : u \in W_0^{1,2}(\Omega),\ \|u\|_{2^*} = 1\}, \end{aligned}$$

the value $S_0(\Omega)$ cannot be attained if Ω is starshaped. The following proposition shows that the same is true for any $\Omega \neq \mathbb{R}^n$. In particular, this means that the solution from Remark 6.3(i) (for $p = p_S$ and $\lambda = 0$) is not a minimizer of $S_0(\cdot, \Omega)$.

Proposition 6.5. *We have $S_0(\Omega_1) = S_0(\Omega_2)$ for any open sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$. If $\Omega \neq \mathbb{R}^n$, then $S_0(\Omega)$ is not attained.*

Proof. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open. Since $S_0(\Omega) = S_0(x + \Omega)$ for any $x \in \mathbb{R}^n$, we may assume $0 \in \Omega_1 \cap \Omega_2$. Denote $w^R(x) := w(Rx)$.

Let $\varepsilon > 0$ and $0 \neq u_1 \in W_0^{1,2}(\Omega_1)$, $S_0(u_1, \Omega_1) < S_0(\Omega_1) + \varepsilon$. Setting $\tilde{u}_1(x) := u(x)$ if $x \in \Omega_1$, $\tilde{u}_1(x) = 0$ if $x \notin \Omega_1$, we have $\tilde{u}_1 \in W_0^{1,2}(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$ and $\operatorname{supp}(\tilde{u}_1^R) \subset \Omega_2$ if R is sufficiently large. Let u_2 be the restriction of $\tilde{u}_1^R$ to Ω_2. Then $u_2 \in W_0^{1,2}(\Omega_2)$, $u_2 \neq 0$, and

$$\begin{aligned} S_0(\Omega_2) \leq S_0(u_2, \Omega_2) &= S_0(\tilde{u}_1^R, \mathbb{R}^n) = S_0(\tilde{u}_1, \mathbb{R}^n) \\ &= S_0(u_1, \Omega_1) < S_0(\Omega_1) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain $S_0(\Omega_2) \leq S_0(\Omega_1)$. Exchanging the role of Ω_1 and Ω_2 we obtain the reversed inequality.

Now assume $\Omega \neq \mathbb{R}^n$, $u \in W_0^{1,2}(\Omega)$ and $S_0(u, \Omega) = S_0(\Omega)$. We may assume $u \geq 0$, $u \neq 0$. Set $\tilde{u}(x) := u(x)$ for $x \in \Omega$, $\tilde{u}(x) := 0$ otherwise. Then $S_0(\tilde{u}, \mathbb{R}^n) = S_0(\Omega) = S_0(\mathbb{R}^n)$, hence $\tilde{u}$ is a minimizer of $S_0(\cdot, \mathbb{R}^n)$ and the proof of Theorem 6.2 shows the existence of $\mu > 0$ such that $\tilde{u}$ is a classical positive solution of the equation $-\Delta u = \mu|u|^{p-1}u$ in $\mathbb{R}^n$. But this is a contradiction with $u = 0$ outside Ω. □

Remark 6.6. Best constant in Sobolev's inequality. The function $S_0(\cdot, \mathbb{R}^n)$ attains its minimum $S := S_0(\mathbb{R}^n) = \left(n(n-2)\pi\right)^{-1/2}\left(\Gamma(n)/\Gamma(n/2)\right)^{1/n}$ at any function of the form $u_\varepsilon(x - x_0)$, where $\varepsilon > 0$, $x_0 \in \mathbb{R}^n$ and

$$u_\varepsilon(x) := (\varepsilon^2 + |x|^2)^{-(n-2)/2}.$$

This was proved by symmetrization techniques in [43] and [508] (for more general results of this kind see [111] and the references therein). If we set

$$C_\varepsilon := \left[n(n-2)\varepsilon^2\right]^{(n-2)/4},$$

then the functions $C_\varepsilon u_\varepsilon(\cdot - x_0)$ are the only positive classical solutions of (6.1) with $\Omega = \mathbb{R}^n$, $p = p_S$ and $\lambda = 0$: This follows from Theorems 8.1 and 9.1 below. □

Theorem 6.7. *Let $n \geq 3$ and $p = p_S$. Assume Ω bounded, $0 < \lambda < \lambda_1$. Let S be the constant from Remark* 6.6.

(i) *If $S_\lambda(\Omega) < S$, then there exists $u \in W_0^{1,2}(\Omega)$ such that $u > 0$ in Ω and $S_\lambda(\Omega) = S_\lambda(u, \Omega)$.*

(ii) *If λ is close to λ_1, then $S_\lambda(\Omega) < S$.*

Proof. (i) Let $\{u_k\}$ be a minimizing sequence for $S_\lambda(\cdot, \Omega)$, $\|u_k\|_{2^*} = 1$. Replacing u_k by $|u_k|$ we may assume $u_k \geq 0$. Since

$$\left(1 - \frac{\lambda}{\lambda_1}\right)\int_\Omega |\nabla u_k|^2\,dx \leq \int_\Omega \left[|\nabla u_k|^2 - \lambda u_k^2\right] dx = S_\lambda(u_k, \Omega) \to S_\lambda(\Omega),$$

the sequence $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$ and we may assume $u_k \rightharpoonup u$ in $W_0^{1,2}(\Omega)$. Due to the embeddings $W_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and $W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ we obtain $u_k \rightharpoonup u$ in $L^{2^*}(\Omega)$ and $u_k \to u$ in $L^2(\Omega)$. Passing to a subsequence we may assume $u_k(x) \to u(x)$ a.e. Given $t \in [0, 1]$, denote

$$\psi_k = \psi_k(t) := 2^*\left(u_k + (t-1)u\right)\left|u_k + (t-1)u\right|^{2^*-2}, \qquad \psi = \psi(t) := 2^* tu|tu|^{2^*-2}.$$

Then $\psi_k \to \psi$ a.e. in Ω and ψ_k, ψ are uniformly bounded in $L^{2_*}(\Omega)$, where $2_* := (2^*)' = 2n/(n+2)$. Using Vitali's convergence theorem we obtain

$$\int_\Omega \left[|u_k|^{2^*} - |u_k - u|^{2^*}\right] dx = \int_\Omega \int_0^1 \frac{d}{dt}|u_k + (t-1)u|^{2^*} \,dt\,dx$$
$$= \int_0^1 \int_\Omega \psi_k u \,dx\,dt \to \int_0^1 \int_\Omega \psi u \,dx\,dt = \int_\Omega |u|^{2^*}\,dx \quad \text{as } k \to \infty,$$

hence

$$\|u\|_{2^*}^{2^*} = 1 - \|u_k - u\|_{2^*}^{2^*} + o(1),$$

where $o(1) \to 0$ as $k \to \infty$. The weak convergence $u_k \rightharpoonup u$ in $W_0^{1,2}(\Omega)$ implies

$$\int_\Omega |\nabla u_k|^2\,dx = \int_\Omega |\nabla(u_k - u)|^2\,dx + \int_\Omega |\nabla u|^2\,dx + o(1),$$

hence

$$\begin{aligned} S_\lambda(\Omega) &= S_\lambda(u_k, \Omega) + o(1) \\ &= \int_\Omega |\nabla(u_k-u)|^2\,dx + \int_\Omega \left[|\nabla u|^2 - \lambda u^2\right] dx + o(1) \\ &\geq S\|u_k - u\|_{2^*}^2 + S_\lambda(\Omega)\|u\|_{2^*}^2 + o(1) \\ &\geq S\|u_k - u\|_{2^*}^{2^*} + S_\lambda(\Omega)\|u\|_{2^*}^{2^*} + o(1) \\ &= \big(S - S_\lambda(\Omega)\big)\|u_k - u\|_{2^*}^{2^*} + S_\lambda(\Omega) + o(1). \end{aligned}$$

Now $S > S_\lambda(\Omega)$ implies $u_k \to u$ in $L^{2^*}(\Omega)$, hence $\|u\|_{2^*} = 1$. The weak lower semi-continuity of the norm in $W_0^{1,2}(\Omega)$ guarantees

$$S_\lambda(u,\Omega) \leq \liminf_{k\to\infty} S_\lambda(u_k, \Omega) = S_\lambda(\Omega),$$

thus $S_\lambda(u,\Omega) = S_\lambda(\Omega)$. Similarly as in the proof of Theorem 6.2, a suitable positive multiple of u is a classical positive solution of (6.1) with $p = p_S$, hence $u > 0$ in Ω.

(ii) Let φ_1 be the first eigenfunction, $\|\varphi_1\|_{2^*} = 1$. Then

$$S_\lambda(\Omega) \leq S_\lambda(\varphi_1, \Omega) = (\lambda_1 - \lambda)\int_\Omega \varphi_1^2\,dx < S$$

if λ is close to λ_1. □

Corollary 6.8. *Let $n \geq 3$ and $p = p_S$. Assume Ω bounded, $0 < \lambda < \lambda_1$. If λ is close to λ_1, then problem (6.1) has a classical positive solution.*

Remarks 6.9. (i) **Positive solutions in the critical case** [98]. Let Ω be bounded, $p = p_S$,

$$\lambda^* := \inf\{\lambda \in (0,\lambda_1) : S_\lambda(\Omega) < S\}.$$

Set $u_\varepsilon(x) := (\varepsilon + |x|^2)^{-(n-2)/2}$ (cf. Remark 6.6) and assume $0 \in \Omega$. If $n \geq 4$ and $\lambda > 0$, then careful estimates show $S_\lambda(u_\varepsilon\varphi, \Omega) < S$ provided $\varphi \in \mathcal{D}(\Omega)$ is nonnegative, $\varphi = 1$ in a neighborhood of 0 and ε is small enough. Consequently, $\lambda^* = 0$ in this case and problem (6.1) possesses a positive solution for any $\lambda \in (0,\lambda_1)$.

Now let $n = 3$ and $\Omega = B_1(0)$. If $\lambda > \lambda_1/4$, then $S_\lambda(u_\varepsilon\varphi,\Omega) < S$ provided $\varphi(x) = \cos(\pi|x|/2)$ and ε is small enough. On the other hand, one can use a Pohozaev-type identity for radial functions in order to prove that (6.1) does not have positive radial solutions if $\lambda \leq \lambda_1/4$. Since any positive solution of (6.1) is symmetric due to [239] we have $\lambda^* = \lambda_1/4$ in this case and the problem possesses positive solutions if and only if $\lambda \in (\lambda_1/4, \lambda_1)$.

Another proof of the above results for $\Omega = B_1(0)$ based on the ODE techniques can be found in [40]. The authors use the symmetry of positive solutions $u = u(|x|)$ of (6.1) and the substitution $y(t) = u(|x|)$, $t = (n-2)^{n-2}|x|^{-(n-2)}$, which transforms the problem into the ODE $y'' + t^{-k}(\lambda y + y^{p_S}) = 0$ with $k := 2(n-1)/(n-2)$.

(ii) **Uniqueness for $p \leq p_S$.** Uniqueness of positive solutions of (6.1) in the case $\Omega = B_1(0)$, $p \leq p_S$, was established in [239] (if $\lambda = 0$), [393] (if $\lambda \geq 0$, $p \leq p_{sg}$), [310] (if $\lambda < 0$, $p < p_S$) and [544], [503] (if $\lambda > 0$, $p \leq p_S$). Some of these articles contain also uniqueness results for more general functions $f(|x|, u)$ and for Ω being an annulus.

Uniqueness fails for general bounded domains (see (iii) and (iv) below). On the other hand if Ω satisfies some convexity and symmetry properties, then uniqueness (and non-degeneracy) for positive solutions of (6.1) is true, at least for some values of p and/or λ (see [147], [118], [256], for example).

(iii) **Nonradial minimizers.** Let $\Omega = \{x : 1 < |x| < 2\}$, $n \geq 3$, $\lambda = 0$ and $p > 1$. Set

$$\begin{aligned} S(u,\Omega,p) &:= \frac{\int_\Omega |\nabla u|^2\,dx}{\|u\|_{p+1}^2}, \\ S(\Omega,p) &:= \inf\{S(u,\Omega,p) : u \in W_0^{1,2}(\Omega)\ u \neq 0\}, \\ S^r(\Omega,p) &:= \inf\{S(u,\Omega,p) : u \in W_0^{1,2}(\Omega)\ u \neq 0,\ u \text{ is radial}\}. \end{aligned}$$

By Remark 6.3(i), problem (6.1) with $\lambda = 0$ has a positive radially symmetric solution u which minimizes $S(\cdot,\Omega,p)$ in the class of radial functions. Since $S(\Omega,p_S)$ is not attained (see Proposition 6.5), we have $S(\Omega,p_S) < S^r(\Omega,p_S)$. It is easy to see that the functions $p \mapsto S(\Omega,p)$ and $p \mapsto S^r(\Omega,p)$ are continuous. Consequently, $S(\Omega,p) < S^r(\Omega,p)$ also for $p < p_S$, p close to p_S. Since $S(\Omega,p)$ is attained in the subcritical case, the corresponding (positive) minimizer is not radially symmetric.

(iv) **Effect of the topology of domain.** Let Ω be bounded, $n \geq 3$, $p = p_S$ and $\lambda = 0$. The above considerations show that (6.1) has a positive solution if Ω is an annulus but it does not possess positive solutions if Ω is starshaped. It was proved in [47] that this problem has positive solutions whenever the homology of dimension d of Ω with $\mathbb{Z}_2$ coefficients is nontrivial for some positive integer d. In particular, this is true when $n = 3$ and Ω is not contractible. On the other hand, there are several examples showing that positive solutions do exist even if Ω is contractible (see [149], for example).

Let Ω be bounded and let its Ljusternik-Schnirelman category be bigger than 1. If $p < p_S$, then problem (6.1) admits multiple positive solutions whenever p is close to p_S or $\lambda < 0$ and $|\lambda|$ is large enough (see [68]); the same is true if $p = p_S$, $\lambda > 0$ is small and $n \geq 4$ (see [456], [323]). Again, this topological condition on Ω is not necessary (see [148], where multiple positive solutions are constructed for any $p < p_S$, $\lambda = 0$ and Ω being starshaped, and see [408] for the critical case).

(v) **Critical case in the unit ball.** Let $\Omega = B_1(0)$, $n \geq 3$, $p = p_S$ and consider radial (classical) solutions of (6.1).

Due to Corollary 5.2, nontrivial solutions do not exist if $\lambda \leq 0$. Denote by X the space of all radial functions in $W_0^{1,2}(\Omega)$ and let λ_k^r denote the k-th eigenvalue of $-\Delta$ in X ($\lambda_k^r = k^2\pi^2$ if $n = 3$). The corresponding radial eigenfunction φ_k^r (considered as a function of $r := |x|$) has $(k-1)$ zeros in $(0,1)$ and each point $(0, \lambda_k^r) \in X \times \mathbb{R}$ is a bifurcation point for (6.1) (see [451]). The corresponding bifurcation branch $\mathcal{B}_k$ of nontrivial solutions is an unbounded continuous curve and u has $(k-1)$ zeros for any $(u, \lambda) \in \mathcal{B}_k$. Moreover, there exists $\mu_k := \lim\{\lambda : (u,\lambda) \in \mathcal{B}_k,\ \|u\|_X \to \infty\}$, $k = 1, 2, \ldots$, and we have $\mu_k = (k - \frac{1}{2})^2\pi^2$ if $n = 3$, $\mu_1 = 0$ if $n \geq 4$, $\mu_{k+1} = \lambda_k^r$ if $n = 4, 5$, $\mu_{k+1} \in (0, \lambda_k^r)$ if $n = 6$, $\mu_k = 0$ if $n \geq 7$ (see Figure 1 and [40], [41], [42], [39]).

Denote $\tilde{\mu}_k := \inf\{\lambda : (u, \lambda) \in \mathcal{B}_k\}$. The results mentioned in (i) and (ii) imply $\tilde{\mu}_1 = \mu_1 = \lambda_1/4$ if $n = 3$, $\tilde{\mu}_1 = \mu_1 = 0$ if $n \geq 4$. Similarly, [34] and [234] imply $\tilde{\mu}_2 = \mu_2$ if $n = 4$, $\tilde{\mu}_2 < \mu_2$ if $n = 5$ but the relation between $\tilde{\mu}_2$ and μ_2 for $n \in \{3, 6\}$ seems to be an open problem.

Denote also $\lambda_* := \inf\{\tilde{\mu}_k : k \geq 2\}$. Then $\lambda_* > 0$ provided $n \leq 6$ (see [39]). On the other hand, problem (6.1) with $\Omega = B_1(0)$, $n \geq 4$, $p = p_S$ and $\lambda > 0$ has infinitely many nontrivial solutions in $W_0^{1,2}(\Omega)$ (see [212]). Consequently, if $n \in \{4, 5, 6\}$ and $\lambda < \lambda_*$, then all these solutions (except for $\pm u_1$ where u_1 denotes the unique positive solution) have to be nonradial. The existence of (nonradial sign-changing) solutions for $\Omega = B_1(0)$, $n = 3$ and $\lambda \in (0, \lambda_1/4]$ seems to be open.

Many interesting results on singular radial solutions of (6.1) for $\Omega = B_1(0)$ and $p > 1$ can be found in [73]. □

Remarks 6.10. Supercritical case. Let $n \geq 3$, $p > p_S$.

(i) If $\lambda = 0$, then the analogue of the result of [47] mentioned in Remark 6.9(iv) does not hold (see [406], [407]).

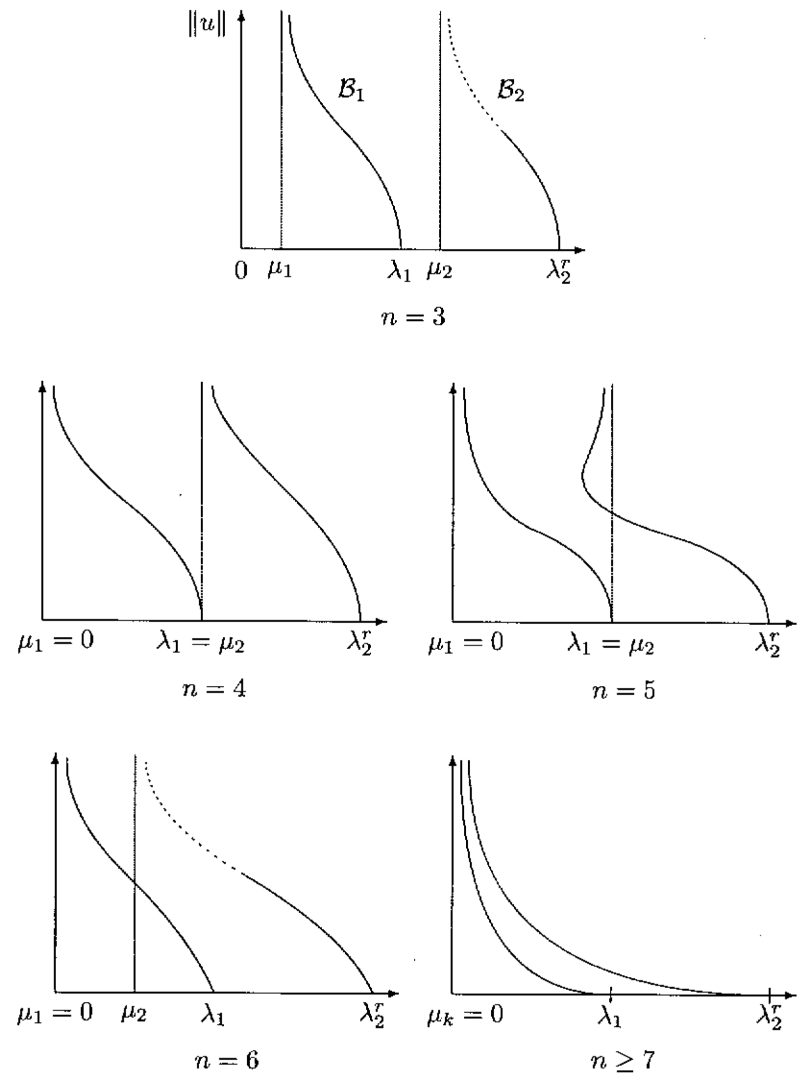

Figure 1: Bifurcation diagrams for radial solutions of (6.1) with $p = p_S$ and $\Omega = B_1(0)$.

(ii) Let $\Omega = B_1(0)$. Then the points $(0, \lambda_k^r)$ from Remark 6.9(v) are bifurcation points for (6.1) also in this case. Let $\mathcal{B}_k(p)$ denote the corresponding bifurcation branch and let $\mu_k(p), \tilde{\mu}_k(p)$ have similar meaning as in Remark 6.9(v). If $n > 6$, assume also $p < p_{ZZ} := \big(n + 1 - \sqrt{2n-3}\big)/\big(n - 3 - \sqrt{2n-3}\big)$. Then

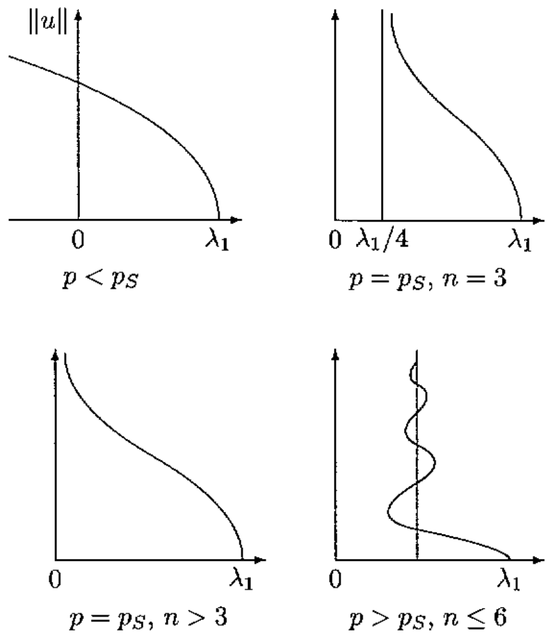

Figure 2: Bifurcation diagrams for positive solutions of (6.1) with $\Omega = B_1(0)$.

$0 < \tilde{\mu}_1(p) < \mu_1(p) < \lambda_1$ and problem (6.1) has infinitely many radial positive (classical) solutions if $\lambda = \mu_1(p)$ (see Figure 2 and [546], [104], [365]). It is not clear whether the condition $p < p_{ZZ}$ for $n > 6$ is optimal, but some restrictions on n or p for this behavior of $\mathcal{B}_1(p)$ may be expected. In fact, bifurcation diagrams for positive solutions of the related problem

$$\left.\begin{aligned} -\Delta u &= \lambda(1+u)^p, && x \in B_1(0), \\ u &= 0, && x \in \partial B_1(0), \end{aligned}\right\} \qquad (6.5)$$

in the supercritical case are completely different for $p < p_{JL}$ and $p \geq p_{JL}$, where p_{JL} is defined in (9.3) (see Figure 3 and [293]).

Note also that the same diagrams as in Figure 3 are true for the problem

$$\left.\begin{aligned} -\Delta u &= \lambda e^u, && x \in B_1(0), \\ u &= 0, && x \in \partial B_1(0), \end{aligned}\right\} \qquad (6.6)$$

and the three cases I, II and III correspond to $n \leq 2$, $3 \leq n \leq 9$ and $n \geq 10$, respectively. □

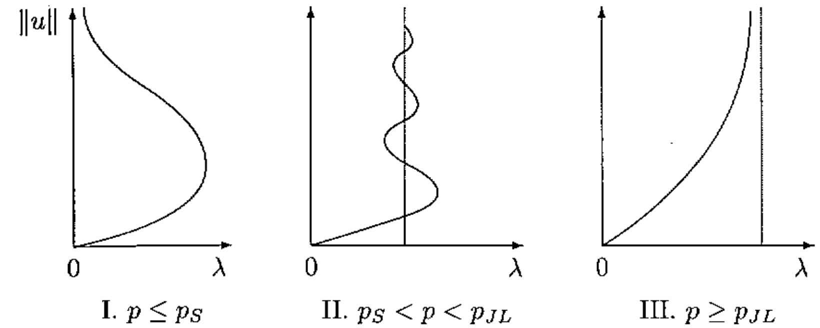

Figure 3: Bifurcation diagrams for positive solutions of (6.5).

7. Minimax methods

In this section we look for saddle points of the energy functional E defined in (3.5) by minimax methods. Throughout this section we assume that f satisfies the growth assumption (3.4) so that E is a C^1-functional in the Hilbert space $W_0^{1,2}(\Omega)$ and its critical points correspond to (variational) solutions of (2.1).

Even if we considered a finite-dimensional space $X = \mathbb{R}^2$ and a smooth functional $E : X \to \mathbb{R}$, then (looking at the graph of E as the earth's surface) existence of a saddle (mountain pass) on a mountain range between two valleys is not clear, in general. For example, if $E : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto e^x - y^2$, $A_0 = (0, -2)$, $A_1 = (0, 2)$, then any path from A_1 to A_2 in $\mathbb{R}^2$ has to cross the line $\{y = 0\}$ where $E > 0 > \max\{E(A_1), E(A_2)\}$, but the functional E does not possess critical points at all. If one looks for a point with a minimal height on the "mountain range" described by the graph of E on $\{(x, y) : y = 0\}$, then any minimizing sequence has the form $(x_k, 0)$, where $x_k \to -\infty$. In particular, it is not compact and we cannot choose a subsequence converging to the desired saddle point. Therefore, dealing with abstract functionals E in a real Banach space X, we shall need additional information on E which will prevent the problem mentioned above.

Definition 7.1. A sequence $\{u_k\}$ in X is called a **Palais-Smale sequence** if the sequence $\{E(u_k)\}$ is bounded and $E'(u_k) \to 0$. We say that E satisfies condition (PS) if any Palais-Smale sequence is relatively compact. We say that E satisfies condition $(\mathrm{PS})_\beta$ (Palais-Smale condition at level β) if any sequence $\{u_k\}$ satisfying $E(u_k) \to \beta$, $E'(u_k) \to 0$, is relatively compact. A real number β is called a **critical value** of E if there exists $u \in X$ with $E'(u) = 0$ and $E(u) = \beta$. □

The following mountain pass theorem is due to [23]. Our proofs of this theorem and Theorems 7.4, 7.8 below closely follow those in [505, Chapter II].

Theorem 7.2. *Suppose that $E \in C^1(X)$ satisfies* (PS). *Let $u_0, u_1 \in X$,*

$$\begin{aligned} M &:= \max\{E(u_0), E(u_1)\}, \\ P &:= \{p \in C([0,1], X) : p(0) = u_0,\ p(1) = u_1\}, \\ \beta &:= \inf_{p\in P} \max_{t\in[0,1]} E(p(t)). \end{aligned} \tag{7.1}$$

If $\beta > M$, then β is a critical value of E.

Given $\beta \in \mathbb{R}$ and $\delta > 0$, denote

$$N_\delta = N_\delta(\beta) := \{u \in X : |E(u) - \beta| \le \delta,\ \|E'(u)\| \le \delta\}$$

and $E_\beta := \{u \in X : E(u) < \beta\}$.

In the proof of Theorem 7.2 we shall need the following deformation lemma.

Lemma 7.3. *Suppose that $E \in C^1(X)$ and let $N_\delta(\beta) = \emptyset$ for some $\delta < 1$. Choose $\varepsilon = \delta^2/2$. Then there exists a continuous mapping $\Phi : X \times [0,1] \to X$ such that*
(i) *$\Phi(u,t) = u$ whenever $t = 0$ or $|E(u) - \beta| \ge 2\varepsilon$,*
(ii) *$t \mapsto E(\Phi(u,t))$ is nonincreasing for all u,*
(iii) *$\Phi(E_{\beta+\varepsilon}, 1) \subset E_{\beta-\varepsilon}$.*
In addition, $\Phi(\cdot, t)$ is odd if E is even.

Proof. In order to avoid all technicalities we shall assume, in addition, that $E \in C^2(X)$ and X is a Hilbert space. Notice that these assumptions are satisfied in our applications if f has the form (2.2), for example (and see e.g. [505] for the proof in the general case).

Choose functions $\varphi, \psi : \mathbb{R} \to [0,1]$ such that φ is smooth, $\varphi(t) = 1$ for $|t-\beta| \le \varepsilon$, $\varphi(t) = 0$ for $|t - \beta| \ge 2\varepsilon$, $\psi(t) = 1$ for $t \le 1$ and $\psi(t) = 1/t$ for $t > 1$. The vector field

$$\mathcal{F} : X \to X : u \mapsto -\varphi\big(E(u)\big)\psi\big(\|E'(u)\|\big)\nabla E(u)$$

is bounded and locally Lipschitz. Consequently, the initial value problem

$$\begin{aligned} \Phi_t(u,t) &= \mathcal{F}\big(\Phi(u,t)\big), \qquad \text{for } t \in [0,1], \\ \Phi(u,0) &= u \end{aligned}$$

has a unique solution for any $u \in X$. The function Φ defined in this way is obviously continuous and satisfies (i). Denoting $v := \Phi(u,t)$ we have

$$\frac{d}{dt}E\big(\Phi(u,t)\big) = \frac{d}{dt}E(v) = E'(v)\mathcal{F}(v) = -\varphi\big(E(v)\big)\psi\big(\|E'(v)\|\big)\|E'(v)\|^2 \le 0,$$

thus (ii) is true.

Assertion (iii) will be proved by a contradiction argument. Let $u \in E_{\beta+\varepsilon}$ and assume $\Phi(u,1) \notin E_{\beta-\varepsilon}$. Then (ii) implies $|E(\Phi(u,t)) - \beta| \le \varepsilon < \delta$ for $t \in [0,1]$, hence $N_\delta = \emptyset$ implies $\|E'(\Phi(u,t))\| \ge \delta$ for $t \in [0,1]$. Using this estimate and the properties of the functions φ, ψ we get

$$\begin{aligned} E\big(\Phi(u,1)\big) &= E(u) + \int_0^1 \frac{d}{dt}E(\Phi(u,t))\,dt \\ &= E(u) - \int_0^1 \underbrace{\varphi(\dots)}_{=1}\underbrace{\psi(\dots)\|E'(\Phi(u,t))\|^2}_{\ge\delta^2}\,dt \\ &< \beta + \varepsilon - \delta^2 \le \beta - \varepsilon, \end{aligned}$$

a contradiction. □

Proof of Theorem 7.2. Assume that β is not a critical value of E. Then it is easy to use condition (PS) in order to find $\delta > 0$ such that $N_\delta(\beta) = \emptyset$. We may assume $\delta < 1$, $\delta^2 < \beta - M$. Let $\varepsilon := \frac{1}{2}\delta^2$ be from Lemma 7.3. By the definition of β there exists $p \in P$ such that $\max_{t\in[0,1]} E(p(t)) < \beta + \varepsilon$. Since $E(u_i) \le M < \beta - \delta^2 = \beta - 2\varepsilon$ for $i = 0, 1$, Lemma 7.3(i) guarantees that $p_1 : t \mapsto \Phi(p(t), 1)$ is an element of P. Now Lemma 7.3(iii) implies $\max_{t\in[0,1]} E(p_1(t)) \le \beta - \varepsilon$, which contradicts the definition of β. □

The next theorem is again due to [23]. It represents a symmetric variant of Theorem 7.2 and we will use it for the proof of existence of infinitely many solutions of problem (2.1).

Theorem 7.4. *Suppose that $E \in C^1(X)$ is even and satisfies* (PS). *Let X^+, X^- be closed subspaces of X with $\dim X^- = \operatorname{codim} X^+ + 1 < \infty$. Let $E(0) = 0$ and let there exist $\alpha, \rho, R > 0$ such that $E(u) \ge \alpha$ for all $u \in S_\rho^+ := \{u \in X^+ : \|u\| = \rho\}$ and $E(u) \le 0$ for all $u \in X^-$, $\|u\| \ge R$. Set*

$$\begin{aligned} \Gamma &:= \{h \in C(X,X) : h \text{ is odd},\ h(u) = u \text{ if } E(u) \le 0\}, \\ \beta &:= \inf_{h\in\Gamma} \max_{u\in X^-} E\big(h(u)\big). \end{aligned}$$

Then β is a critical value of E, $\beta \ge \alpha$.

The proof of the above theorem will be almost the same as the proof of Theorem 7.2 provided we prove the following Intersection Lemma.

Lemma 7.5. *If $\rho > 0$ and $h \in \Gamma$, then $h(X^-) \cap S_\rho^+ \ne \emptyset$.*

Proof of Theorem 7.4. Lemma 7.5 implies $\beta \ge \alpha$. Assume that β is not a critical value of E. Then $N_\delta(\beta) = \emptyset$ for some $\delta > 0$ and we may assume $\delta < 1$, $\delta^2 < \alpha$. Let $\varepsilon := \delta^2/2$ and Φ be from Lemma 7.3 and choose $h \in \Gamma$ such that

$E(h(u)) < \beta + \varepsilon$ for all $u \in X^-$. Set $h_1(u) := \Phi(h(u), 1)$. Then $h_1 \in \Gamma$ and $E(h_1(u)) = E(\Phi(h(u), 1)) < \beta - \varepsilon$, due to Lemma 7.3(iii). But this contradicts the definition of β. □

In the proof of Lemma 7.5 we shall need the notion of **Krasnoselskii genus**.

Definition 7.6. Let $\mathcal{A}$ be the set of all closed subsets of X satisfying $A = -A$. If $A \in \mathcal{A}$, then we set $\gamma(A) := 0$ if $A = \emptyset$ and

$$\gamma(A) := \inf\{m : \exists h \in C(A, \mathbb{R}^m \setminus \{0\}),\ h \text{ odd}\}$$

otherwise. □

The following proposition is proved in [505, Propositions II.5.2 and II.5.4]:

Proposition 7.7. *Suppose that $A, A_1, A_2 \in \mathcal{A}$ and $h \in C(X, X)$ is odd. Then the following is true:*
(1) $\gamma(A) \geq 0$, $\gamma(A) = 0$ *if and only if* $A = \emptyset$,
(2) *if* $A_1 \subset A_2$, *then* $\gamma(A_1) \leq \gamma(A_2)$,
(3) $\gamma(A_1 \cup A_2) \leq \gamma(A_1) + \gamma(A_2)$,
(4) $\gamma(A) \leq \gamma(\overline{h(A)})$,
(5) *if A is compact and $0 \notin A$, then $\gamma(A) < \infty$ and there exists a symmetric neighborhood U of A such that $\overline{U} \in \mathcal{A}$ and $\gamma(A) = \gamma(\overline{U})$.*
(6) *Let D be a bounded symmetric neighborhood of zero in Y, where Y is a subspace of X with $m := dim(Y) < \infty$, and let $\partial_Y D$ denote the boundary of D in Y. Then $\gamma(\partial_Y D) = m$.*

Proof of Lemma 7.5. Let $\rho > 0$ and $h \in \Gamma$. Set $R_1 := \max\{R, \rho\}$, $B^-_{R_1} := \{u \in X^- : \|u\| < R_1\}$ and $S_\rho := \{u \in X : \|u\| = \rho\}$. Since $E(u) \leq 0$ for $u \in X^-$, $\|u\| \geq R$, we have $\|h(u)\| = \|u\| > \rho$ for all $u \in X^-$, $\|u\| > R_1$, hence $h(X^-) \cap S_\rho = h(\overline{B^-_{R_1}}) \cap S_\rho$ is compact. In particular, $A := h(X^-) \cap S^+_\rho$ fulfills the assumptions of Proposition 7.7(5), thus there exists its symmetric neighborhood U with $\gamma(\overline{U}) = \gamma(A)$. By (2) and (3) in Proposition 7.7 we obtain

$$\gamma(A) = \gamma(\overline{U}) \geq \gamma(h(X^-) \cap S_\rho \cap \overline{U}) \geq \gamma(h(X^-) \cap S_\rho) - \gamma(B), \tag{7.2}$$

where $S_\rho := \{u \in X : \|u\| = \rho\}$ and $B := h(X^-) \cap S_\rho \setminus U$. Let Z be a direct complement of X^+ in X and let $\pi : X \to Z$ denote the projection along X^+. Since U is a neighborhood of $h(X^-) \cap S^+_\rho$, we get $B \cap X^+ = \emptyset$, hence $0 \notin \pi(B)$ and the definition of γ implies

$$\gamma(B) \leq \dim Z = \operatorname{codim} X^+. \tag{7.3}$$

Now (2) and (4) in Proposition 7.7 guarantee $\gamma(h(X^-) \cap S_\rho) \geq \gamma(h^{-1}(S_\rho) \cap X^-)$. Since $h(0) = 0$ and $h(u) = u$ for $u \in X^-$, $\|u\| > R$, the set $h^{-1}(S_\rho) \cap X^-$ contains the relative boundary of $\{u \in X^- : \|h(u)\| < \rho\}$ which is a symmetric bounded neighborhood of zero in X^-. Consequently, using (2) and (6) in Proposition 7.7 we arrive at

$$\gamma(h(X^-) \cap S_\rho) \geq \dim X^- = \operatorname{codim} X^+ + 1. \tag{7.4}$$

Now (7.2)–(7.4) imply $\gamma(A) \geq 1$, hence $A \neq \emptyset$. □

Theorems 7.2 and 7.4 guarantee the following solvability result.

Theorem 7.8. *Assume Ω bounded. Let f be a Carathéodory function, and let there exist $p < p_S$, $R > 0$ and $\mu > 2$ such that $|f(x, u)| \leq C(1 + |u|^p)$ for all $x \in \Omega$, $u \in \mathbb{R}$ and $f(x, u)u \geq \mu F(x, u) > 0$ for all $x \in \Omega$ and $|u| > R$.*
(i) *If there exist $c < \lambda_1$ and $\rho \in (0, 1)$ such that $f(x, u)/u \leq c$ for all $x \in \Omega$ and $|u| < \rho$, then there exists a positive solution of* (2.1).
(ii) *If $f(x, -u) = -f(x, u)$ for all $x \in \Omega$ and $u \in \mathbb{R}$, then there exists a sequence $\{u_k\}$ of solutions of* (2.1) *with $E(u_k) \to \infty$ as $k \to \infty$.*

Proof. The energy functional E associated with (2.1) is C^1. Let us first verify that E satisfies condition (PS). Let $\{u_k\}$ be a Palais-Smale sequence. Denote $|u|_{1,2} := \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}$ and notice that this is an equivalent norm in $X := W^{1,2}_0(\Omega)$. Then

$$\begin{aligned} o(1 + |u_k|_{1,2}) &= -E'(u_k)u_k = -|u_k|^2_{1,2} + \int_\Omega f(x, u_k)u_k\,dx \\ &= \left(\frac{\mu}{2} - 1\right)|u_k|^2_{1,2} + \int_\Omega [f(x, u_k)u_k - \mu F(x, u_k)]\,dx - \mu E(u_k) \\ &\geq \left(\frac{\mu}{2} - 1\right)|u_k|^2_{1,2} - C_1, \end{aligned}$$

where $C_1 > 0$ is independent of k. Consequently, the sequence $\{u_k\}$ is bounded in X. We have $\nabla E(u) = u + \mathcal{F}_1(u)$, where $\mathcal{F}_1$ is compact.[2] Since $\{u_k\}$ is bounded in X, we may assume (passing to a subsequence if necessary) $\mathcal{F}_1(u_k) \to w$ in X for some $w \in X$. Since $o(1) = \nabla E(u_k) = u_k - \mathcal{F}_1(u_k)$, we obtain $u_k \to w$, hence $\{u_k\}$ is relatively compact.

(i) We will use Theorem 7.2. In order to get a positive solution, let us define $\tilde{f}(x, u) := f(x, u)$ if $u \geq 0$, $\tilde{f}(x, u) = 0$ otherwise, $\tilde{F}(x, u) := \int_0^u \tilde{f}(x, s)\,ds$, $\tilde{E}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega \tilde{F}(x, u)\,dx$, and notice that $\tilde{E}$ is C^1 and satisfies condition (PS). Set $u_0 := 0$, then $\tilde{E}(u_0) = 0$. The assumption $f(x, u)/u \leq c$ for $|u| < \rho$ guarantees $|\tilde{F}(x, u)| \leq (c/2)u^2$ for $|u| < \rho$. If $|u| \geq \rho$, then the growth assumption $|f(x, u)| \leq C(1 + |u|^p)$ implies

$$|\tilde{F}(x, u)| \leq C(|u| + |u|^{p+1}) \leq (c/2)u^2 + C_2|u|^{p+1},$$

[2] The Nemytskii mapping $\mathcal{F} : L^{p+1}(\Omega) \to L^{(p+1)'}(\Omega) : u \mapsto f(\cdot, u)$ is continuous. The embedding $I_p : X \hookrightarrow L^{p+1}(\Omega)$ is compact, hence the dual mapping $I'_p : (L^{p+1}(\Omega))' \to X'$ is compact as well. Let $R : X' \to X$ denote the Riesz isomorphism in the Hilbert space X (thus $RE'(u) = \nabla E(u)$) and let $J : L^{(p+1)'}(\Omega) \to (L^{p+1}(\Omega))'$ be the isomorphism defined by $(Jw)u = \int_\Omega uw\,dx$ for $u \in L^{p+1}(\Omega)$. Then $\nabla E(u) = u + \mathcal{F}_1(u)$, where $\mathcal{F}_1 : X \to X : u \mapsto RI'_p J\mathcal{F} I_p(u)$ is compact.

where $C_2 := C(1+\rho^{-p})$. Consequently, if C_p denotes the norm of the embedding $X \hookrightarrow L^{p+1}(\Omega)$, then

$$\tilde{E}(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{c}{2}\int_\Omega u^2\,dx - C_2\int_\Omega |u|^{p+1}\,dx$$
$$\geq \Big(\frac{1}{2} - \frac{c}{2\lambda_1} - C_2 C_p^{p+1}|u|_{1,2}^{p-1}\Big)|u|_{1,2}^2 \geq \alpha > 0$$

provided $|u|_{1,2} = \delta$ is small enough. Now the assumption $f(x,u)u \geq \mu F(x,u) > 0$ implies $\frac{d}{du}\big(u^{-\mu}F(x,u)\big) \geq 0$ for $u > R$, hence $F(x,u) \geq b(x)u^\mu$ for $u > R$, where $b(x) := R^{-\mu}F(x,R) > 0$. Hence, fixing $u \in X$, $u > 0$ in Ω, denoting

$$A(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx, \qquad B(u) := \int_\Omega b(x)u^\mu\,dx > 0, \tag{7.5}$$

and taking $t > 0$, we obtain

$$\tilde{E}(tu) = E(tu) \leq t^2 A(u) - t^\mu B(u) + C_3 \to -\infty \quad \text{as } t \to \infty,$$

where we used the estimate

$$\int_{0<tu\leq R}\big[b(x)(tu)^\mu - F(x,tu)\big]\,dx \leq C_3$$

with C_3 independent of t and u. Hence, choosing $u_1 := tu$ with t large enough we have $E(u_1) < 0$. Let β be the number defined in Theorem 7.2. Since any path joining u_0 and u_1 has to intersect the sphere $S_\delta := \{u : |u|_{1,2} = \delta\}$, we have $\beta \geq \alpha > 0$ and Theorem 7.2 guarantees the existence of a solution u with $\tilde{E}(u) \geq \alpha$. Since $\tilde{f}(x,u) = 0$ for $u \leq 0$, the maximum principle implies $u \geq 0$. Now $E(u) = \tilde{E}(u) > 0$, hence $u \neq 0$ and using the maximum principle again we obtain $u > 0$ in Ω.

(ii) Choose a positive integer k. Let X^- denote the linear hull of $\varphi_1, \varphi_2, \dots, \varphi_k$, and X^+ be the closure of the linear hull of $\varphi_k, \varphi_{k+1}, \dots$. The growth condition on f guarantees $|F(x,u)| \leq C_F(1+|u|^{p+1})$ for suitable $C_F > 0$. Set $q := p_S$ if $n \geq 3$ and choose $q > p$ otherwise. Let $C_4 := C_F C_q^{p+1-r}$ and $C_5 := C_F|\Omega|$, where C_q denotes the norm of the embedding $I_q : W_0^{1,2}(\Omega) \hookrightarrow L^{q+1}(\Omega)$, $r \in (0,p+1)$ is defined by $r/2 + (p+1-r)/q = 1$ and $|\Omega|$ denotes the measure of Ω. If $u \in X^+$ and $\|u\| = \rho := \rho_k := \big(\lambda_k^{r/2}/(4C_4)\big)^{1/(p-1)}$, then

$$E(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - C_F\int_\Omega |u|^{p+1}\,dx - C_5$$
$$\geq \frac{1}{2}|u|_{1,2}^2 - C_F\|u\|_2^r\|u\|_{q+1}^{p+1-r} - C_5$$
$$\geq \Big(\frac{1}{2} - C_4\lambda_k^{-r/2}|u|_{1,2}^{p-1}\Big)|u|_{1,2}^2 - C_5$$
$$= \Big(\frac{1}{2} - C_4\lambda_k^{-r/2}\rho^{p-1}\Big)\rho^2 - C_5 = C_6\lambda_k^{r/(p-1)} - C_5,$$

where $C_6 = (4^p C_4)^{-1/(p-1)}$. Denote $\alpha = \alpha_k := \inf\{E(u) : u \in X^+,\ |u|_{1,2} = \rho\}$. Since $\lambda_k \to \infty$ as $k \to \infty$, we have $\alpha_k \to \infty$.

On the other hand, estimates in (i) show $E(tu) \leq t^2A(u) - t^\mu B(u) - C_3$, where A, B are defined in (7.5). Since $A(u) = 1/2$ for $|u|_{1,2} = 1$ and $C_7 := \inf\{B(u) : u \in X^-,\ |u|_{1,2} = 1\} > 0$, we have

$$E(u) \leq \frac{1}{2}|u|_{1,2}^2 - C_7|u|_{1,2}^\mu + C_3 \quad \text{for all } u \in X^-,$$

hence the assumptions of Theorem 7.4 are satisfied for any k large enough and we obtain a sequence of critical points u_k of E satisfying $E(u_k) \geq \alpha_k$. (In fact, a more careful choice of ρ above enables one to use Theorem 7.4 for any k.) □

Remarks 7.9. (i) **Linking.** Let f be differentiable in u, $f(x,0) = 0$, $f(x,u)/u \geq f_u(x,0)$ for all $x \in \Omega$ and $u \in \mathbb{R}$. If the assumption $f(x,u)/u \leq c < \lambda_1$ for u small in Theorem 7.8(i) fails, then one can use a modification of the mountain pass theorem, so called "linking", in order to prove the existence of a nontrivial solution of (2.1) (see [505, Section II.8] and the references therein).

(ii) **Perturbation results.** Consider the problem

$$\left.\begin{aligned} -\Delta u &= |u|^{p-1}u + \varphi, && x \in \Omega,\\ u &= 0, && x \in \partial\Omega, \end{aligned}\right\} \tag{7.6}$$

where $\Omega \subset \mathbb{R}^n$ is bounded, $1 < p < p_S$ and $\varphi \in W^{-1,2}(\Omega) := \big(W_0^{1,2}(\Omega)\big)'$. Theorem 7.8(ii) guarantees existence of infinitely many solutions of (7.6) provided $\varphi = 0$. The same result is known to be true for φ belonging to a residual set in $W^{-1,2}(\Omega)$ (see [45]) and for all $\varphi \in W^{-1,2}(\Omega)$ provided $p(n-2) < n$ (see [300, Théorème V.4.6.]; see also [506], [46], [452] and [48]). On the other hand, if $n > 2$, $p \in [p_{sg}, p_S)$ and φ is a general (smooth) function, then even the solvability of (7.6) seems to be open.

(iii) **Unbounded domains.** If $\Omega = \mathbb{R}^n$, then the existence of infinitely many solutions of (2.1) is known in many cases as well. We refer to [76], [140], [139], [9] and the references therein.

(iv) **Critical case.** Let $\Omega \subset \mathbb{R}^n$ be bounded, $p = p_S$ and $\lambda > 0$. If $n \geq 7$, then problem (6.1) possesses infinitely many solutions, see [162]. Such a result is known for any $n \geq 4$ if the domain Ω exhibits suitable symmetries (see [212]) but not for general domains (cf. also the results for Ω being a ball mentioned in Remark 6.9(v)). If $n = 6$ and $\lambda \in (0,\lambda_1)$, then (6.1) has at least two (pairs of) solutions for any bounded Ω, see [117]. Recall also that if $\lambda \leq 0$, $p \geq p_S$ and Ω is starshaped, then (6.1) does not possess nontrivial classical solutions due to Corollary 5.2. □

8. Liouville-type results

In order to prove a priori bounds for positive solutions of (2.1) with $f(x,u) \sim u^p$ as $u \to +\infty$, $1 < p < p_S$ (see the rescaling method in Section 12), it will be important to know that the problems

$$-\Delta u = u^p, \qquad x \in \mathbb{R}^n \tag{8.1}$$

and

$$\left.\begin{aligned} -\Delta u &= u^p, & x &\in \mathbb{R}^n_+, \\ u &= 0, & x &\in \partial\mathbb{R}^n_+, \end{aligned}\right\} \tag{8.2}$$

do not possess positive bounded (classical) solutions. Here $\mathbb{R}^n_+$ denotes the half-space $\{x \in \mathbb{R}^n : x_n > 0\}$. In fact, we shall see in Chapter II that these Liouville-type results have important applications for parabolic problems as well. In this section we even prove that these problems do not possess any positive classical solution. The following two results are due to [240], [241], except for Theorem 8.1(ii) which was proved in [108].

Theorem 8.1. *Let $\Omega = \mathbb{R}^n$ and $p > 1$.*

(i) *If $p < p_S$, then equation* (8.1) *does not possess any positive classical solution.*

(ii) *If $p = p_S$, then any positive classical solution of* (8.1) *is radially symmetric with respect to some point.*

Theorem 8.2. *Let $1 < p \le p_S$. Then problem* (8.2) *does not possess any positive classical solution.*

We will see in the next section that the condition $p < p_S$ is optimal for nonexistence in $\mathbb{R}^n$. However, in the case of a half-space and if we consider only *bounded* positive solutions, nonexistence is known for a larger range of exponents, namely $p < p_S' := (n+1)/(n-3)_+$ (note that p_S' is the Sobolev exponent in $(n-1)$ dimensions). This result is due to [150].

Theorem 8.3. *Let $1 < p < p_S'$, where*

$$p_S' := \begin{cases} \infty & \text{if } n \le 3, \\ (n+1)/(n-3) & \text{if } n > 3. \end{cases}$$

Then problem (8.2) *does not possess any positive, bounded classical solution.*

On the other hand, under a stronger assumption on p, one can extend the nonexistence result in $\mathbb{R}^n$ to elliptic inequalities. The following result is due to [238].

Theorem 8.4. *Let $1 < p \le p_{sg}$. Then the inequality*

$$-\Delta u \ge u^p, \qquad x \in \mathbb{R}^n \tag{8.3}$$

does not possess any positive classical solution.

Remarks 8.5. (i) It seems unknown if the condition $p \le p_S$ is optimal for the nonexistence of positive solutions of (8.2). In the case of positive bounded solutions, the results recently announced in [178] indicate that the condition $p < p_S'$ can be improved, but the optimal exponent seems to remain unknown.

(ii) The condition $p \le p_{sg}$ in Theorem 8.4 is optimal, as shown by the explicit example $u(x) = k(1+|x|^2)^{-1/(p-1)}$ with $n \ge 3$, $p > p_{sg}$ and $k > 0$ small enough.

(iii) Consider the inequality $-\Delta u \ge u^p$ in the half-space $\mathbb{R}^n_+$ (no boundary conditions required). Then nonexistence of positive solutions holds whenever $p \le (n+1)/(n-1)$ (see [74]).

(iv) Consider "quasi-solutions" of (8.1), i.e. (nonnegative) functions satisfying the double inequality

$$au^p \le -\Delta u \le u^p, \quad x \in \mathbb{R}^n, \tag{8.4}$$

for some $a \in (0,1)$. It is shown in [509] that if $1 < p < p_S$ and $a \in (0,1)$ is close enough to 1, then (8.4) has no positive solution $u \in C^2(\mathbb{R}^n)$ (see also Remark 8.8(ii)). On the other hand, if $p > p_{sg}$ and $a \in (0,1)$ is small enough, then (8.4) possesses positive solutions $u \in C^2(\mathbb{R}^n)$. Note that a simple example is provided by the function $u(x) = k(1+|x|^2)^{-1/(p-1)}$ with $k > 0$ large enough. □

We start by proving Theorem 8.4, which is much easier than Theorems 8.1 and 8.2. The following proof (cf. [74], [501], [372]) is based on a rescaled test-function argument, and it is different and simpler than the original proof of [238].

Proof of Theorem 8.4. Take $\xi \in \mathcal{D}(B_1)$, $0 \le \xi \le 1$, with $\xi = 1$ for $|x| \le 1/2$, and let $m = 2p/(p-1)$. Fix $R > 0$ and define $\varphi_R(x) = \xi^m(x/R)$. We observe that

$$\Delta\varphi_R = mR^{-2}\big[\xi^{m-1}\Delta\xi + (m-1)\xi^{m-2}|\nabla\xi|^2\big](x/R)$$

hence

$$|\Delta\varphi_R| \le CR^{-2}\xi^{m-2}(x/R)\chi_{\{|x|>R/2\}} = CR^{-2}\varphi_R^{1/p}\chi_{\{|x|>R/2\}}.$$

Multiplying (8.3) by φ_R, integrating by parts, and using Hölder's inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} u^p\varphi_R \le -\int_{\mathbb{R}^n} u\,\Delta\varphi_R &\le CR^{-2}\int_{R/2<|x|<R} u\,\varphi_R^{1/p} \\ &\le CR^{n(p-1)/p-2}\Big(\int_{R/2<|x|<R} u^p\varphi_R\Big)^{1/p}. \end{aligned} \tag{8.5}$$

In particular, it follows that

$$\int u^p \varphi_R \le C R^{n-2p/(p-1)}. \tag{8.6}$$

If $p < p_{sg}$, i.e. $n - 2p/(p-1) < 0$, then this implies $u \equiv 0$ upon letting $R \to \infty$. If $p = p_{sg}$, then (8.6) implies $\int_{\mathbb{R}^n} u^p < \infty$. Therefore, the RHS of (8.5) goes to 0 as $R \to \infty$ and we again conclude that $u \equiv 0$. ☐

Theorem 8.1 is much more delicate. Note that, in view of Theorem 8.4, we may restrict ourselves to $n \ge 3$. We will give a first proof of Theorem 8.1(i) which, like the original proof of [240], is based on integral estimates for (local) positive solutions (cf. Proposition 8.6 below). Here we essentially follow the (simplified) treatment of [83]. Next, we will prove Theorem 8.2 by using moving plane arguments, following [241]. We will then give a second, completely different proof of Theorem 8.1(i), also based on moving planes arguments, which is due to [123], [78] and allows us to prove Theorem 8.1(ii) at the same time. We point out that the techniques of both proofs of Theorem 8.1(i) are important and can be extended to some other problems (see e.g. Section 21 and [123], [78], respectively). Finally, we will prove Theorem 8.3 following [150]. Note that, although the proofs of both Theorems 8.2 and 8.3 are based on moving planes arguments, they use different ideas: reduction to the one-dimensional problem on a half-line for the former, monotonicity and reduction to the $(n-1)$-dimensional problem in the whole space for the latter.

Proposition 8.6. *Let $1 < p < p_S$ and let B_1 be the unit ball in $\mathbb{R}^n$. There exists $r = r(n,p) > \max(n(p-1)/2, p)$ such that if $0 < u \in C^2(B_1)$ is a solution of*

$$-\Delta u = u^p \tag{8.7}$$

in B_1, then

$$\int_{|x|<1/2} u^r \le C(n,p). \tag{8.8}$$

Let us assume for the moment that Proposition 8.6 is proved and deduce some consequences of it. To prove Theorem 8.1(i) it suffices to apply a simple homogeneity argument.

Proof of Theorem 8.1(i). Assume that u is a positive solution of (8.1). Then, for each $R > 0$, $v(x) := R^{2/(p-1)} u(Rx)$ solves (8.7) in B_1. It follows from Proposition 8.6 that

$$\begin{aligned}\int_{|y|<R/2} u^r(y)\,dy &= R^n \int_{|x|<1/2} u^r(Rx)\,dx \\ &= R^{n-2r/(p-1)} \int_{|x|<1/2} v^r(x)\,dx \le C(n,p) R^{n-2r/(p-1)}.\end{aligned}$$

By letting $R \to \infty$, we conclude that $\int_{\mathbb{R}^n} u^r(y)\,dy = 0$, a contradiction. ☐

As another important consequence of Proposition 8.6, we have the following result (cf. [151]) concerning singularities of local solutions to (8.7) in arbitrary domains. Note that when $p_{sg} < p < p_S$, the upper estimate in Theorem 4.1 concerning isolated singularities follows as a special case.

Theorem 8.7. *Let $1 < p < p_S$ and let Ω be an arbitrary domain in $\mathbb{R}^n$. There exists $C = C(n,p) > 0$ such that if $0 < u \in C^2(\Omega)$ is a solution of*

$$-\Delta u = u^p, \qquad x \in \Omega, \tag{8.9}$$

then

$$u(x) \le C(n,p)[\mathrm{dist}(x, \partial\Omega)]^{-2/(p-1)}. \tag{8.10}$$

Proof. It relies on Proposition 8.6 and a bootstrap argument. Let

$$r > \max(n(p-1)/2, p)$$

be given by Proposition 8.6. We may fix $\rho > 1$ such that

$$p - \frac{1}{\rho} < \frac{2r}{n}. \tag{8.11}$$

Assume that v is a solution of

$$-\Delta v = v^p \quad \text{in } B := \{x \in \mathbb{R}^n : |x| < 1\}. \tag{8.12}$$

Let i be a nonnegative integer and assume that, for all $\omega \subset\subset B$, there exists a constant $C_i(n,p,\omega) > 0$ (independent of v) such that

$$\|v\|_{L^{r\rho^i}(\omega)} \le C_i(n,p,\omega). \tag{8.13}$$

Note that (8.13) is true for $i = 0$ by Proposition 8.6. Since $r\rho^i/p > 1$ and

$$\frac{p}{r\rho^i} - \frac{1}{r\rho^{i+1}} = \frac{1}{r\rho^i}\Big(p - \frac{1}{\rho}\Big) < \frac{2}{n}$$

due to (8.12), we may apply Proposition 47.6(ii) to deduce that (8.13) is true with i replaced by $i+1$. After a finite number of steps, we obtain $\|v\|_{L^k(\omega)} \le C(n,p,\omega)$ for some $k > n/2$. We may then apply Proposition 47.6(ii) once more to deduce that

$$v(0) \le C(n,p). \tag{8.14}$$

Now assume that u is a solution of (8.9), fix $x_0 \in \Omega$ and let $R := \mathrm{dist}(x_0, \partial\Omega)$. Then $v(x) := R^{2/(p-1)} u(x_0 + Rx)$ solves (8.12) and the conclusion follows from (8.14). ☐

Remarks 8.8. (i) **More general nonlinearities.** Results similar to Theorem 8.7 for more general nonlinearities can be found in [240], [83], [473], [424]. In particular, universal singularity estimates of the type of (8.10) are established in [424] when the nonlinearity u^p is replaced by any $f(x,u)$ such that $f(x,u) \sim u^p$, as $u \to \infty$, with $1 < p < p_S$. The method of proof is different: The estimate is directly deduced from the Liouville-type Theorem 8.1(i) by using rescaling and doubling arguments (see Theorem 26.8 and Lemma 26.11 below for a similar approach in the parabolic case).

(ii) **Singularities of quasi-solutions.** For "quasi-solutions" of (8.1) (cf. Remark 8.5(iv)), the local behavior near an isolated singularity was studied in [509]. Let $\Omega = B(0,1) \setminus \{0\}$. If $p_{sg} < p < p_S$ and $a \in (0,1)$ is close enough to 1, then any positive classical solution of

$$au^p \le -\Delta u \le u^p, \qquad x \in \Omega, \tag{8.15}$$

satisfies $\limsup_{x\to 0} |x|^{2/(p-1)}u(x) < \infty$. On the contrary, if $p > p_{sg}$ and $a \in (0,1)$ is small enough, then there exist solutions of (8.15) with arbitrarily large growth rates as $x \to 0$.

On the other hand, by a straightforward modification of the proof of [424, Theorem 2.1], one can show the following uniform and global property: For each $p \in (1, p_S)$, there exist $a = a(n,p) \in (0,1)$ and $C(n,p) > 0$ such that, for any domain $\Omega \subset \mathbb{R}^n$, estimate (8.10) is true for any positive solution $u \in C^2(\Omega)$ of (8.15). Note that, as a consequence of this estimate, one recovers the nonexistence result in Remark 8.5(iv).

(iii) **Radial supercritical case.** When $p \ge p_S$, $\Omega = B_R$ and u is a radial positive classical solution of (8.9), a similar argument as in Remark 4.3(iii) shows that $u(r) \le C(R-r)^{-2/(p-1)}$, $0 \le r < R$, for some $C > 0$. However the constant C cannot depend only on n, p, since otherwise this would imply nonexistence of radial positive classical solutions of (8.9) for $\Omega = \mathbb{R}^n$ and $p \ge p_S$, hence contradicting Theorem 9.1 below. □

We now turn to the proof of Proposition 8.6. It is based on a key gradient estimate for local solutions of (8.7) (see (8.22) below). To establish this estimate, we prepare the following lemma, which provides a family of integral estimates relating any C^2-function with its gradient and its Laplacian. The proof relies on the Bochner identity (8.18), on the change of variable $v = u^{k+1}$, and on test-functions of the form φv^m.

In the rest of this section, we use the notation $\int = \int_\Omega$ for simplicity.

Lemma 8.9. *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $0 \le \varphi \in \mathcal{D}(\Omega)$, and $0 < u \in C^2(\Omega)$. Fix $q \in \mathbb{R}$ and denote*

$$I = \int \varphi u^{q-2}|\nabla u|^4, \quad J = \int \varphi u^{q-1}|\nabla u|^2 \Delta u, \quad K = \int \varphi u^q (\Delta u)^2.$$

Then, for any $k \in \mathbb{R}$ with $k \ne -1$, there holds

$$\alpha I + \beta J + \gamma K \le \frac{1}{2}\int u^q |\nabla u|^2 \Delta\varphi + \int u^q[\Delta u + (q-k)u^{-1}|\nabla u|^2]\nabla u \cdot \nabla\varphi, \tag{8.16}$$

where

$$\alpha = -\frac{n-1}{n}k^2 + (q-1)k - \frac{q(q-1)}{2}, \quad \beta = \frac{n+2}{n}k - \frac{3q}{2}, \quad \gamma = -\frac{n-1}{n}.$$

Proof. *Step 1.* We first claim that for all $v \in C^2(\Omega)$, $v > 0$ and any $m \in \mathbb{R}$, there holds

$$\begin{aligned}\frac{m(1-m)}{2}\int \varphi v^{m-2}|\nabla v|^4 - \frac{3m}{2}\int \varphi v^{m-1}|\nabla v|^2 \Delta v - \frac{n-1}{n}\int \varphi v^m (\Delta v)^2 \\ \le \frac{1}{2}\int v^m |\nabla v|^2 \Delta\varphi + \int [v^m \Delta v + m v^{m-1}|\nabla v|^2]\nabla v \cdot \nabla\varphi.\end{aligned} \tag{8.17}$$

First note that, by density, it suffices to prove (8.17) for $v \in C^3(\Omega)$. To prove the claim, we start from the identity

$$\frac{1}{2}\Delta|\nabla v|^2 = \nabla(\Delta v)\cdot\nabla v + |D^2 v|^2, \tag{8.18}$$

where $|D^2 u|^2 = \sum_{1\le i,j\le n} (u_{x_i x_j})^2$. Multiplying by φv^m and integrating over Ω, we obtain

$$T_1 + T_2 := \int \varphi v^m \nabla(\Delta v)\cdot\nabla v + \int \varphi v^m |D^2 v|^2 = \frac{1}{2}\int \varphi v^m \Delta|\nabla v|^2 =: T_3. \tag{8.19}$$

Integrating by parts and using $\varphi \in \mathcal{D}(\Omega)$, we can rewrite the first and third terms as follows:

$$\begin{aligned}T_1 &= -\int (\Delta v)\nabla\cdot(\varphi v^m \nabla v) \\ &= -\int v^m (\Delta v)\nabla v\cdot\nabla\varphi - m\int \varphi v^{m-1}|\nabla v|^2 \Delta v - \int \varphi v^m (\Delta v)^2\end{aligned}$$

and

$$\begin{aligned}T_3 &= \int |\nabla v|^2 \Big[\frac{1}{2}v^m \Delta\varphi + m v^{m-1}\nabla v\cdot\nabla\varphi + \frac{m}{2}\varphi(v^{m-1}\Delta v + (m-1)v^{m-2}|\nabla v|^2)\Big] \\ &= \frac{1}{2}\int v^m|\nabla v|^2\Delta\varphi + m\int v^{m-1}|\nabla v|^2 \nabla v\cdot\nabla\varphi \\ &\quad + \frac{m}{2}\int \varphi v^{m-1}|\nabla v|^2 \Delta v + \frac{m(m-1)}{2}\int \varphi v^{m-2}|\nabla v|^4.\end{aligned}$$

Moving the first term of T_1 to the right of (8.19) and the last two terms of T_3 to the left, it follows that

$$\frac{m(1-m)}{2}\int \varphi v^{m-2}|\nabla v|^4 - \frac{3m}{2}\int \varphi v^{m-1}|\nabla v|^2\Delta v + \int \varphi v^m |D^2 v|^2$$
$$= \int \varphi v^m (\Delta v)^2 + \frac{1}{2}\int v^m |\nabla v|^2 \Delta\varphi + \int \big[v^m \Delta v + m v^{m-1}|\nabla v|^2\big]\nabla v\cdot\nabla\varphi. \tag{8.20}$$

By Cauchy-Schwarz' inequality (applied with the inner product $(A,B)=\mathrm{tr}(AB^*)$ on matrices), we have

$$(\Delta v)^2 = \big(\mathrm{tr}(D^2v)\big)^2 \le \mathrm{tr}\,\big[(D^2v)(D^2v)^*\big]\,\mathrm{tr}\,(I_n) = n|D^2 v|^2. \tag{8.21}$$

Due to $\varphi\ge 0$, Claim (8.17) follows by combining (8.20) and (8.21).

Step 2. We set $v=u^{k+1}$, $m=(k+1)^{-1}(q-2k)$, that is $q=(k+1)m+2k$, and we compute

$$\int \varphi v^{m-2}|\nabla v|^4 = (k+1)^4\int \varphi u^{(k+1)(m-2)+4k}|\nabla u|^4 = (k+1)^4 I,$$

$$\int \varphi v^{m-1}|\nabla v|^2\Delta v = (k+1)^3\int \varphi u^{(k+1)(m-1)+3k}|\nabla u|^2(\Delta u + k u^{-1}|\nabla u|^2)$$
$$= (k+1)^3(kI+J),$$

$$\int \varphi v^m(\Delta v)^2$$
$$= (k+1)^2\int \varphi u^{(k+1)m+2k}\big[(\Delta u)^2 + 2k(\Delta u)u^{-1}|\nabla u|^2 + k^2u^{-2}|\nabla u|^4\big]$$
$$= (k+1)^2(k^2 I + 2kJ + K),$$

$$\int v^m(\Delta v)\nabla v\cdot\nabla\varphi = (k+1)^2\int u^{(k+1)m+2k}\big[\Delta u + k u^{-1}|\nabla u|^2\big]\nabla u\cdot\nabla\varphi,$$

and

$$\int v^{m-1}|\nabla v|^2\nabla v\cdot\nabla\varphi = (k+1)^3\int u^{(k+1)m+2k-1}|\nabla u|^2\nabla u\cdot\nabla\varphi.$$

Substituting in (8.17) and dividing by $(k+1)^2$, we get

$$\Big[\frac{m(1-m)}{2}(k+1)^2 - \frac{3m}{2}k(k+1) - \frac{n-1}{n}k^2\Big]I + \Big[-\frac{3m}{2}(k+1) - 2k\frac{n-1}{n}\Big]J$$
$$-\frac{n-1}{n}K \le \frac{1}{2}\int u^q|\nabla u|^2\Delta\varphi + \int u^q\big[\Delta u + (k+m(k+1))u^{-1}|\nabla u|^2\big]\nabla u\cdot\nabla\varphi,$$

which readily implies the lemma. □

Lemma 8.10. (i) *Let Ω be an arbitrary domain in $\mathbb{R}^n$, and $0\le\varphi\in\mathcal{D}(\Omega)$. Let $0<u\in C^2(\Omega)$ be a solution of (8.7) in Ω. Fix $q,k\in\mathbb{R}$ with $q>-p$, $k\ne -1$ and denote*

$$I=\int \varphi u^{q-2}|\nabla u|^4,\quad K=\int\varphi u^{2p+q}.$$

Then there holds

$$\alpha I + \delta K \le \frac{1}{2}\int u^q|\nabla u|^2\Delta\varphi + C\int \big[u^{p+q} + u^{q-1}|\nabla u|^2\big]|\nabla u\cdot\nabla\varphi|, \tag{8.22}$$

where $C=C(n,p,q,k)>0$ and

$$\alpha = -\frac{n-1}{n}k^2 + (q-1)k - \frac{q(q-1)}{2},\quad \delta = \frac{1}{p+q}\Big(\frac{3q}{2} - \frac{n+2}{n}k\Big) - \frac{n-1}{n}. \tag{8.23}$$

(ii) *Assume that $1<p<p_S$. Then there exist $q,k\in\mathbb{R}$, with $q\ne -p$, $k\ne -1$, such that the constants α,δ defined in (8.23) satisfy*

$$\alpha,\delta>0,\qquad 2p+q>n(p-1)/2. \tag{8.24}$$

Proof. (i) Since $-\Delta u = u^p$, we have

$$(p+q)J = -\int \varphi\,(p+q)u^{p+q-1}|\nabla u|^2 = -\int \varphi\,\nabla u\cdot\nabla(u^{p+q})$$
$$= \int \varphi\,(\Delta u)u^{p+q} + \int (\nabla\varphi\cdot\nabla u)u^{p+q},$$

where J is defined in Lemma 8.9, hence

$$(p+q)J = -\int\varphi u^{2p+q} + \int(\nabla\varphi\cdot\nabla u)u^{p+q}.$$

Substituting in (8.16), we obtain (8.22).

(ii) A simple computation shows that $\delta>0$ and $2p+q>n(p-1)/2$ is equivalent to

$$k<k_0(q):=\frac{q}{2}-\frac{(n-1)p}{n+2}\quad\text{and}\quad q>q_0(p):=\frac{(n-4)p-n}{2}.$$

For $k=k_0(q)$, we have

$$\alpha = \alpha(k_0(q)) = \frac{n-1}{n}\Big(-\frac{q^2}{4} + \frac{(n-1)pq}{n+2} - \frac{(n-1)^2p^2}{(n+2)^2}\Big)$$
$$+(q-1)\Big(\frac{q}{2} - \frac{(n-1)p}{n+2}\Big) - \frac{q(q-1)}{2}$$
$$= \frac{n-1}{n}\Big(-\frac{q^2}{4} + \frac{(n-1)pq}{n+2} - \frac{(n-1)^2p^2}{(n+2)^2} - \frac{np(q-1)}{n+2}\Big)$$
$$= \frac{n-1}{n}\Big(-\frac{q^2}{4} - \frac{pq}{n+2} + \frac{n(n+2)p - (n-1)^2p^2}{(n+2)^2}\Big).$$

The discriminant of the above polynomial in q is given by

$$D = \frac{p^2 + n(n+2)p - (n-1)^2p^2}{(n+2)^2} = \frac{np[(n+2)-(n-2)p]}{(n+2)^2} > 0.$$

Therefore we have $\alpha(k_0(q)) > 0$ for $q \in (-\frac{2p}{n+2} - 2\sqrt{D}, -\frac{2p}{n+2} + 2\sqrt{D})$. Moreover, $-\frac{2p}{n+2} > q_0(p)$ is equivalent to $n(n+2) > (n^2-2n-4)p$, which is true due to $p < p_S$. Choosing

$$q = -\frac{2p}{n+2} \quad \text{and} \quad k = k_0(q)^- = \Big(-\frac{np}{n+2}\Big)^- \quad (\text{with } k \neq -1),$$

we see that (8.24) is fulfilled. □

Proof of Proposition 8.6. Take q, k as in Lemma 8.10(ii) and $\Omega = B_1$. We shall estimate the terms on the RHS of (8.22). Let $\xi \in \mathcal{D}(\Omega)$, be such that $\xi = 1$ for $|x| \le 1/2$ and $0 \le \xi \le 1$. Put $\theta = (3p+1+2q)/2(2p+q) \in (0,1)$. By taking $\varphi = \xi^m$ with $m = 2/(1-\theta)$, we have

$$|\nabla\varphi| \le C\xi^{m-1} \le C\varphi^\theta, \quad |\Delta\varphi| \le C\xi^{m-2} \le C\varphi^\theta. \tag{8.25}$$

Fix $\varepsilon > 0$. Using Young's inequality under the form

$$xyz \le \varepsilon x^a + \varepsilon y^b + C(\varepsilon)z^c, \qquad a^{-1} + b^{-1} + c^{-1} = 1,$$

and (8.25), we obtain

$$\int u^q|\nabla u|^2\Delta\varphi = \int \Big(\varphi^{1/2}u^{(q-2)/2}|\nabla u|^2\Big)\Big(\varphi^{(q+2)/2(2p+q)}u^{(q+2)/2}\Big)$$
$$\times\Big(\varphi^{-(p+1+q)/(2p+q)}\Delta\varphi\Big) \le \varepsilon\int \varphi u^{q-2}|\nabla u|^4 + \varepsilon\int \varphi u^{2p+q} + C(\varepsilon),$$

$$C\int u^{p+q}|\nabla u\cdot\nabla\varphi| \le \int\Big(\varphi^{1/4}u^{(q-2)/4}|\nabla u|\Big)\Big(\varphi^{(4p+3q+2)/4(2p+q)}\,u^{(4p+3q+2)/4}\Big)$$
$$\times\Big(\varphi^{-(3p+1+2q)/2(2p+q)}|\nabla\varphi|\Big) \le \varepsilon\int\varphi u^{q-2}|\nabla u|^4 + \varepsilon\int\varphi u^{2p+q} + C(\varepsilon),$$

and

$$C\int u^{q-1}|\nabla u|^2|\nabla u\cdot\nabla\varphi| \le \int\Big(\varphi^{3/4}u^{3(q-2)/4}|\nabla u|^3\Big)\Big(\varphi^{(q+2)/4(2p+q)}\,u^{(q+2)/4}\Big)$$
$$\times\Big(\varphi^{-(3p+1+2q)/2(2p+q)}|\nabla\varphi|\Big) \le \varepsilon\int\varphi u^{q-2}|\nabla u|^4 + \varepsilon\int\varphi u^{q+2p} + C(\varepsilon).$$

Combining this with (8.22), we obtain

$$\alpha I + \delta K \le C(n,p,q,k)\varepsilon(I+K) + C(\varepsilon).$$

Since $\alpha, \delta > 0$, by choosing ε sufficiently small, we conclude that $I, K \le C$, hence (8.8) with $r = 2p + q > \max(n(p-1)/2, p)$. □

Proof of Theorem 8.2. Let u be a positive solution of (8.2).

Assume $n \ge 2$ and denote $x' = (x_1, \dots, x_{n-1})$. Choose $\bar{x}, \tilde{x} \in \mathbb{R}^n_+$ with $\bar{x}_n = \tilde{x}_n$. We will show $u(\bar{x}) = u(\tilde{x})$ so that u depends only on x_n.

Choose the origin to be the point $\big((\frac{\bar{x}+\tilde{x}}{2})', 0\big)$. Given $x \in \overline{\mathbb{R}^n_+}$, set

$$z = \frac{x + e_n}{|x+e_n|^2}, \qquad v(z) = |x+e_n|^{n-2}u(x) = \frac{u(x)}{|z|^{n-2}}.$$

The function v is the **Kelvin transform** of u. It solves the problem

$$\left.\begin{aligned} \Delta v + |z|^\gamma v^p &= 0 \quad &&\text{in } D,\\ v &= 0 &&\text{on } \partial D\setminus\{0\}, \end{aligned}\right\} \tag{8.26}$$

where $D := B_{1/2}(e_n/2)$ and $\gamma := (n-2)p - (n+2) \le 0$. We want to show that v is axisymmetric about the z_n axis, i.e. $v = v(|z'|, z_n)$. Choose any direction e perpendicular to the z_n-axis. Without loss of generality we may assume $e = e_1$.

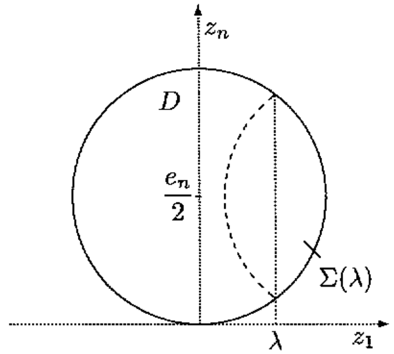

Figure 4: Moving planes.

We shall apply the **moving planes method** to problem (8.26). Given $\lambda \in [0, 1/2)$, set $\Sigma(\lambda) := \{z \in D : z_1 > \lambda\}$, $z^\lambda := (2\lambda - z_1, z_2, \dots, z_n)$. The point z^λ is the **reflection** of z with respect to the hyperplane $\{z_1 = \lambda\}$ and $\Sigma(\lambda)$ is called a **cap**. We next define

$$w(z;\lambda) := v(z^\lambda) - v(z) \qquad \text{for } z \in \overline{\Sigma}(\lambda)$$

(the parameter λ will be omitted in w when no risk of confusion arises). Then

$$\begin{aligned}\Delta w &= \Delta v(z^\lambda) - \Delta v(z) = -|z^\lambda|^\gamma v^p(z^\lambda) + |z|^\gamma v^p(z) \\ &= \big(|z|^\gamma - |z^\lambda|^\gamma\big)v^p(z^\lambda) - |z|^\gamma\big(v^p(z^\lambda) - v^p(z)\big).\end{aligned}$$

Since $v^p(z^\lambda) - v^p(z) = p\xi^{p-1}w(z;\lambda)$ for some $\xi = \xi(z,\lambda)$ lying between $v(z^\lambda)$ and $v(z)$, we obtain

$$\Delta w + |z|^\gamma p\xi^{p-1}w = \big(|z|^\gamma - |z^\lambda|^\gamma\big)v^p(z^\lambda) \geq 0 \quad \text{in } \Sigma(\lambda)\ .$$

The maximum principle (see Proposition 52.1) implies $v > 0$ in D and $\partial v/\partial\nu < 0$ on $\partial D \setminus \{0\}$, hence $w \geq 0$ on $\Sigma(\lambda)$ for λ close to $1/2$.

Set

$$\bar\mu := \inf\{\mu > 0 : w \geq 0 \text{ in } \Sigma(\lambda) \text{ for all } \lambda \geq \mu\}$$

and assume $\bar\mu > 0$. Then $w \geq 0$ on $\Sigma(\bar\mu)$ and there exist $\lambda_i \in (0,\bar\mu)$, $\lambda_i \to \bar\mu$, such that $\inf\{w(z;\lambda_i) : z \in \Sigma(\lambda_i)\} < 0$. Since $w(\cdot;\lambda_i) \geq 0$ on $\partial\Sigma(\lambda_i)$, this infimum is attained at some $q_i \in \Sigma(\lambda_i)$ and $\nabla w(q_i,\lambda_i) = 0$. Since $w(\cdot;\lambda_i) \geq 0$ in an ε-neighborhood of $\partial D \cap \overline{\Sigma(\lambda_i)}$ (with ε being independent of i), we may assume $q_i \to \bar q \in \overline{\Sigma(\bar\mu)} \setminus \partial D$. Continuity arguments and $w \geq 0$ on $\Sigma(\bar\mu)$ guarantee $w(\bar q;\bar\mu) = 0$ and $\nabla w(\bar q;\bar\mu) = 0$, hence $w(\cdot;\bar\mu) \equiv 0$ by the maximum principle. This contradicts $w > 0$ on $\{z \in \partial\Sigma(\bar\mu) : z_1 > \bar\mu\}$. Consequently, $\bar\mu = 0$ and $w(\cdot;0) \geq 0$ on $\Sigma(0)$. A symmetric argument shows $w(\cdot;0) \leq 0$ on $\Sigma(0)$, hence v is symmetric with respect to the hyperplane $\{e_1 = 0\}$. Since this holds for any hyperplane containing the z_n-axis, v is axially symmetric. Hence, $u = u(|x'|, x_n)$ and, consequently, $u(\bar x) = u(\tilde x)$.

Thus we have reduced the problem to the case $n = 1$. Assume that u is a positive solution of

$$\begin{aligned} u''(t) + u^p(t) &= 0, \qquad t > 0, \\ u(0) &= 0. \end{aligned}$$

Since u is concave and positive for $t > 0$, it must fulfill $u' \geq 0$. Fix $t_1 > 0$ and consider $t > t_1$. Then

$$u(t) = u(t_1) + (t - t_1)u'(t_1) + \int_{t_1}^t (t-s)u''(s)\,ds.$$

Since $u''(s) = -u^p(s) \leq -u^p(t_1)$, we obtain

$$0 < u(t) \leq u(t_1) + (t-t_1)u'(t_1) - \frac{1}{2}(t-t_1)^2u^p(t_1),$$

hence

$$u^p(t_1) < \frac{2u(t_1)}{(t-t_1)^2} + \frac{2u'(t_1)}{t-t_1} \to 0 \qquad \text{as } t \to +\infty,$$

a contradiction. $\square$

We now turn to the proof of Theorem 8.1 based on moving planes.

Proof of Theorem 8.1. Due to Theorem 8.4, we may assume $n \geq 3$. Let $p \leq p_S$ and let u be a positive classical solution of (8.1). Set

$$v(z) := \frac{1}{|z|^{n-2}}u\Big(\frac{z}{|z|^2}\Big), \qquad z \in \mathbb{R}^n \setminus \{0\}$$

(v is the Kelvin transform of u). We have $v \in C(\mathbb{R}^n \setminus \{0\})$, $v > 0$,

$$v(z) \leq C|z|^{2-n} \qquad \text{as } |z| \to \infty, \tag{8.27}$$

and v solves the equation

$$\Delta v + |z|^\gamma v^p = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \tag{8.28}$$

where $\gamma := (n-2)p - (n+2) \leq 0$. Due to (8.28) and $n \geq 3$, we infer from Lemma 4.4 that $\Delta v \leq 0$ in $\mathcal{D}'(\mathbb{R}^n)$. It follows from the maximum principle in Proposition 52.3(ii) that, for each $R > 0$,

$$v \geq \eta(R) := \min_{|x|=R} v > 0 \quad \text{in } B_R(0) \setminus \{0\}. \tag{8.29}$$

Given $\lambda \leq 0$, set $z^\lambda := (2\lambda - z_1, z_2, \dots, z_n)$, $\Sigma(\lambda) := \{z \in \mathbb{R}^n : z_1 < \lambda\}$, $\Sigma'(\lambda) := \Sigma(\lambda) \setminus \{0^\lambda\}$ and

$$w(z;\lambda) := v(z^\lambda) - v(z), \quad z \in \overline{\Sigma(\lambda)} \setminus \{0^\lambda\}$$

(the parameter λ will be omitted in w when no risk of confusion arises). As in the preceding theorem we obtain

$$\Delta w + |z|^\gamma p\xi^{p-1}w \leq 0 \quad \text{in } \Sigma'(\lambda), \tag{8.30}$$

where $\xi = \xi(z,\lambda)$ lies between $v(z^\lambda)$ and $v(z)$. Set $\alpha := (n-2)/2$ and $\tilde w(z;\lambda) = |z|^\alpha w(z;\lambda)$. Then

$$\Delta\tilde w - \frac{n-2}{|z|^2}z \cdot \nabla\tilde w + c(z,\lambda)\tilde w \leq 0 \quad \text{in } \Sigma'(\lambda), \tag{8.31}$$

where

$$c(z,\lambda) := -\frac{(n-2)^2}{4|z|^2} + |z|^\gamma p\xi^{p-1}(z,\lambda).$$

Let us first show that

$$\tilde w \geq 0 \quad \text{in } \Sigma'(\lambda), \quad \text{for } \lambda \ll -1. \tag{8.32}$$

We shall argue by contradiction. Assume that $\lambda_i \to -\infty$ and $\inf_{\Sigma'(\lambda_i)} \tilde{w}(\cdot;\lambda_i) < 0$. By (8.27) and (8.29) with $R = 1$, we have $\tilde{w}(z;\lambda_i) \geq 0$ if $|z - 0^{\lambda_i}| < 1$ and i is large enough. Since also, for each i,

$$\tilde{w}(z;\lambda_i) = 0 \text{ on } \partial\Sigma(\lambda_i) \quad \text{and} \quad \tilde{w}(z;\lambda_i) \to 0, \ |z| \to \infty, \tag{8.33}$$

we see that the infimum of $\tilde{w}(\cdot;\lambda_i)$ over $\Sigma'(\lambda_i)$ is attained at some $q_i \in \Sigma'(\lambda_i)$ and $|q_i - 0^{\lambda_i}| \geq 1$. We have $|q_i| \to \infty$, thus $v(q_i) \to 0$. If the sequence $\{q_i^{\lambda_i}\}$ were bounded, then (8.29) would imply $v(q_i^{\lambda_i}) \geq c_1 > 0$, hence $w(q_i) > 0$ for i large, a contradiction. Therefore $|q_i^{\lambda_i}| \to \infty$. Now the definition of v implies $v(z)|z|^{n-2} \to u(0)$ if $|z| \to \infty$, so that we cannot have $|q_i^{\lambda_i}|/|q_i| \to 0$ (otherwise $w(q_i) > 0$ for large i). Thus both $v(q_i)$ and $v(q_i^{\lambda_i})$ can be estimated above by Cq_i^{2-n} for some fixed $C > 0$ and the same is true for $\xi(q_i, \lambda_i)$. Hence,

$$c(q_i, \lambda_i) \leq -\frac{(n-2)^2}{4q_i^2} + \frac{Cp}{q_i^4} < 0 \qquad \text{for } i \text{ large enough.} \tag{8.34}$$

Since $\tilde{w} = \tilde{w}(\cdot;\lambda_i)$ attains an interior minimum at q_i, we have $\Delta\tilde{w}(q_i) \geq 0$, $\nabla\tilde{w}(q_i) = 0$ and $\tilde{w}(q_i) < 0$ so that (8.31) and (8.34) yield a contradiction. This proves (8.32).

Now denote

$$\bar{\mu} := \sup\{\mu \leq 0 : \tilde{w}(\cdot;\lambda) \geq 0 \text{ in } \Sigma'(\lambda) \text{ for all } \lambda \leq \mu\}$$

and assume $\bar{\mu} < 0$. Then $\tilde{w}(\cdot,\bar{\mu}) \geq 0$ in $\Sigma'(\bar{\mu})$ by continuity, and there exist $\lambda_i > \bar{\mu}$, $\lambda_i \to \bar{\mu}$, such that $\inf\{\tilde{w}(z;\lambda_i) : z \in \Sigma'(\lambda_i)\} < 0$. Assume that $\tilde{w}(\cdot,\bar{\mu})$ is not identically zero. Since $\Delta w(\cdot,\bar{\mu}) \leq 0$ in $\Sigma'(\bar{\mu})$, the maximum principle (see Proposition 52.1) implies $w(\cdot,\bar{\mu}) > 0$ in $\Sigma'(\bar{\mu})$. Arguing similarly as for (8.29), we deduce that $w(\cdot,\bar{\mu}) \geq c_2 > 0$ in $U := B_{\bar{\mu}/2}(0^{\bar{\mu}}) \setminus \{0^{\bar{\mu}}\}$. Due to the continuity of v in U and

$$w(z;\lambda_i) = w\big(z - 2(\lambda_i - \bar{\mu})e_1;\bar{\mu}\big) + v\big(z - 2(\lambda_i - \bar{\mu})e_1\big) - v(z), \qquad e_1 := (1,0,\dots,0),$$

we obtain $w(\cdot;\lambda_i) \geq 0$ (hence $\tilde{w}(\cdot;\lambda_i) \geq 0$) in $B_{\bar{\mu}/4}(0^{\lambda_i}) \setminus \{0^{\lambda_i}\}$ for i large. Consequently, in view of (8.33), the infimum of $\tilde{w}(\cdot;\lambda_i)$ over $\Sigma'(\lambda_i)$ has to be attained at some $q_i \in \Sigma'(\lambda_i)$, with $|q_i - 0^{\lambda_i}| \geq \bar{\mu}/4$. Assume $|q_i| \to \infty$. Then $|q_i^{\lambda_i}|/|q_i| \to 1$ and we obtain a contradiction as above (cf. (8.34)). Therefore we may assume that $\{q_i\}$ is bounded and $q_i \to \bar{q} \in \overline{\Sigma(\bar{\mu})} \setminus \{0^{\bar{\mu}}\}$. By continuity and $\tilde{w}(\cdot,\bar{\mu}) \geq 0$, we obtain $\tilde{w}(\bar{q},\bar{\mu}) = 0$ and $\nabla\tilde{w}(\bar{q},\bar{\mu}) = 0$, hence $w(\bar{q},\bar{\mu}) = 0$ and $\nabla w(\bar{q},\bar{\mu}) = 0$. Applying the maximum principle in Proposition 52.1(ii) and (iii) to equation (8.30), it follows that $w(\cdot,\bar{\mu}) \equiv 0$, hence $\tilde{w}(\cdot,\bar{\mu}) \equiv 0$, a contradiction. Consequently, $\tilde{w}(\cdot,\bar{\mu}) \equiv 0$, which means that v is symmetric with respect to $\{z_1 = \bar{\mu}\}$. Now using (8.28) we see that $(-\Delta v)/v^p = |z|^\gamma$ has the same symmetry, which is not possible unless $p = p_S$.

If $p < p_S$, then we get $\bar{\mu} = 0$, so that $w(\cdot,0) \geq 0$ and $v(z^0) \geq v(z)$ provided $z_1 \leq 0$. Considering the function $\tilde{v}(z) := v(z^0)$ instead of v we obtain the reversed inequality, hence $v(z_1, z_2, \dots, z_n) = v(-z_1, z_2, \dots, z_n)$. Repeating this procedure with any given direction instead of e_1 we see that v, hence u, are radially symmetric (about zero). If we repeat this procedure with $\tilde{u}(x) = u(x - x_0)$ for a given $x_0 \neq 0$ instead of u, we show that u is radially symmetric about the point x_0. Since this is true for any x_0, the function u has to be constant. But the only constant solution of (8.1) is the trivial solution.

If $p = p_S$ and $\bar{\mu} < 0$, then v is symmetric with respect to $\{z_1 = \bar{\mu}\}$. If $\bar{\mu} = 0$, then we can repeat the procedure with $\tilde{v}(z) := v(z^0)$ and in any case we obtain the symmetry of v with respect to $\{z_1 = \tilde{\mu}\}$ for suitable $\tilde{\mu}$. Now we can repeat the above proof with directions $e_2, e_3, \dots, e_n$ instead of e_1 and we obtain the existence of $\bar{z} \in \mathbb{R}^n$ such that v is symmetric with respect to $\{z_k = \bar{z}_k\}$ for $k = 1, 2, \dots, n$, hence $v(\bar{z} + z) = v(\bar{z} - z)$ for all z. Rotating the coordinate system and repeating the procedure we find $\tilde{z} \in \mathbb{R}^n$ such that $v(\tilde{z}+z) = v(\tilde{z}-z)$ for all z. Assume $\bar{z} \neq \tilde{z}$. Without loss of generality, we may assume $\bar{z} \neq 0$. The symmetry relations for v imply

$$v(\bar{z}) = v(2\tilde{z} - \bar{z}) = v(3\bar{z} - 2\tilde{z}) = v(4\tilde{z} - 3\bar{z}) = \cdots \to 0,$$

hence $v(\bar{z}) = 0$, a contradiction. Consequently, $\bar{z} = \tilde{z}$ and we obtain the rotational symmetry of v (hence of u) about $\bar{z}$. □

Proof of Theorem 8.3. Assume that (8.2) admits a positive, bounded classical solution u. As a special case of Theorem 21.10 below (which we shall prove by using moving planes arguments), it follows that u is nondecreasing in x_n:

$$\partial_{x_n} u(x) \geq 0, \qquad x \in \mathbb{R}^n_+.$$

Therefore, for each $x' \in \mathbb{R}^{n-1}$,

$$U(x') := \lim_{x_n \to \infty} u(x', x_n)$$

is well defined and is a bounded positive function. Take now $\varphi \in \mathcal{D}(\mathbb{R}^{n-1})$ and $\psi \in \mathcal{D}(\mathbb{R})$, with $\operatorname{supp}\psi \subset (0,1)$ and $\int_0^1 \psi = 1$. Let $k > 0$. Testing the equation with $\varphi(x')\psi(x_n - k)$, we have

$$\begin{aligned} -\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} u^p\,\varphi(x')\,\psi(x_n - k)\,dx_n\,dx' &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} \varphi(x')\,\psi(x_n - k)\,\Delta u\,dx_n\,dx' \\ &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} u\,\Delta\big(\varphi(x')\,\psi(x_n - k)\big)\,dx_n\,dx', \end{aligned}$$

hence

$$\begin{aligned} -\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} & u^p(x', s + k)\,\varphi(x')\,\psi(s)\,ds\,dx' \\ &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} u(x', s + k)\,\Delta\big(\varphi(x')\,\psi(s)\big)\,ds\,dx'. \end{aligned}$$

By dominated convergence, letting $k \to \infty$, it follows that

$$-\int_{\mathbb{R}^{n-1}} U^p(x')\,\varphi(x')\,dx' = -\int_{\mathbb{R}^{n-1}} U^p(x')\,\varphi(x') \int_0^1 \psi(s)\,ds\,dx'$$
$$= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} U(x')\,\Delta\big(\varphi(x')\,\psi(s)\big)\,ds\,dx'.$$

But the RHS is equal to

$$\int_{\mathbb{R}^{n-1}} U(x')\,\Delta_{x'}\varphi(x')\,dx' \int_0^1 \psi(s)\,ds + \int_{\mathbb{R}^{n-1}} U(x')\,\varphi(x')\,dx' \int_0^1 \psi''(s)\,ds$$
$$= \int_{\mathbb{R}^{n-1}} U(x')\,\Delta_{x'}\varphi(x')\,dx'.$$

It follows that U solves (8.1) in $\mathbb{R}^{n-1}$ in the distribution sense, hence in the classical sense (this is a consequence of the boundedness of U and of Remark 47.4). The result is then a consequence of Theorem 8.1(i). □

9. Positive radial solutions of $\Delta u + u^p = 0$ in $\mathbb{R}^n$

In this section we study positive radial classical solutions of the equation

$$-\Delta u = u^p, \qquad x \in \mathbb{R}^n. \tag{9.1}$$

Since this problem does not possess positive classical solutions if $1 < p < p_S$ due to Theorem 8.1, we restrict ourselves to the case $p \geq p_S$. Consequently, $n \geq 3$.

Positive radial classical solutions of (9.1) can be written in the form $u(x) = U(r)$, where $r = |x|$ and $U \in C^2([0,\infty))$ is a positive classical solution of

$$U'' + \frac{n-1}{r}U' + U^p = 0, \quad r \in (0,\infty), \qquad U'(0) = 0. \tag{9.2}$$

It is easily seen that prescribing initial values $U(0) = \alpha > 0$, $U'(0) = 0$, the equation in (9.2) has a unique solution for r small enough. In fact, this equation can be written in the form $(r^{n-1}U')' = -r^{n-1}U^p$ and, by integration we obtain the equivalent integral equation

$$U(r) = \alpha - \int_0^r \int_0^s \Big(\frac{t}{s}\Big)^{n-1} U^p(t)\,dt\,ds,$$

which can be solved by the Banach fixed point theorem.

Let $U_*(r) = c_p r^{-2/(p-1)}$ be the singular solution defined in (3.9) and set

$$p_{JL} := \begin{cases} +\infty & \text{if } n \leq 10, \\ 1 + 4\frac{n-4+2\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n > 10. \end{cases} \tag{9.3}$$

The main result of this section is the following theorem.

Theorem 9.1. *Let $p \geq p_S$. Given $\alpha > 0$, problem (9.2) possesses a unique positive solution $U_\alpha \in C^2([0,\infty))$ satisfying $U_\alpha(0) = \alpha$. This solution is decreasing and we have*

$$U_\alpha(r) = \alpha U_1(\alpha^{(p-1)/2} r). \tag{9.4}$$

If $p > p_S$, then $r^{2/(p-1)}U_\alpha(r) \to c_p$ as $r \to \infty$. If $p = p_S$, then

$$U_1(r) = \Big(\frac{n(n-2)}{n(n-2)+r^2}\Big)^{(n-2)/2}. \tag{9.5}$$

Let $\alpha_1 > \alpha_2 > 0$. If $p \geq p_{JL}$, then $U_*(r) > U_{\alpha_1}(r) > U_{\alpha_2}(r)$ for all $r > 0$. If $p_S < p < p_{JL}$, then U_{α_1} and U_{α_2} intersect infinitely many times and U_{α_1}, U_* intersect infinitely many times as well. If $p = p_S$, then $U_{\alpha_1}, U_{\alpha_2}$ intersect once and U_{α_1}, U_* intersect twice.

Proof. Using the transformation

$$w(s) = r^{2/(p-1)}U(r), \qquad s = \log r, \tag{9.6}$$

problem (9.2) becomes

$$w'' + \beta w' + w^p - \gamma w = 0, \qquad s \in \mathbb{R}, \tag{9.7}$$

where

$$\beta := \frac{1}{p-1}\big((n-2)p - (n+2)\big) \geq 0, \qquad \gamma := c_p^{p-1} = \frac{2}{(p-1)^2}\big((n-2)p - n\big) > 0,$$

and we are looking for solutions w satisfying $w(s), w'(s) \to 0$ as $s \to -\infty$. Set

$$\mathcal{E}(w) = \mathcal{E}(w,w') := \frac{1}{2}|w'|^2 - \frac{\gamma}{2}w^2 + \frac{1}{p+1}w^{p+1}.$$

Then $\mathcal{E}$ is a Lyapunov functional for (9.7); more precisely,

$$\frac{d}{ds}\mathcal{E}\big(w(s)\big) = -\beta\big(w'(s)\big)^2 \leq 0. \tag{9.8}$$

Denoting $x := w$ and $y := w'$, problem (9.7) can be written in the form

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ -\beta y - x^p + \gamma x \end{pmatrix} =: F(x,y) \tag{9.9}$$

where $x > 0$ and $(x,y) \to (0,0)$ as $s \to -\infty$. Problem (9.9) possesses two equilibria, $(0,0)$ and $(c_p, 0)$ lying in the half-space $\{(x,y) : x \geq 0\}$. Denote

$$A_1 := \nabla F(0,0) = \begin{pmatrix} 0 & 1 \\ \gamma & -\beta \end{pmatrix}, \qquad A_2 := \nabla F(c_p,0) = \begin{pmatrix} 0 & 1 \\ -\gamma(p-1) & -\beta \end{pmatrix}.$$

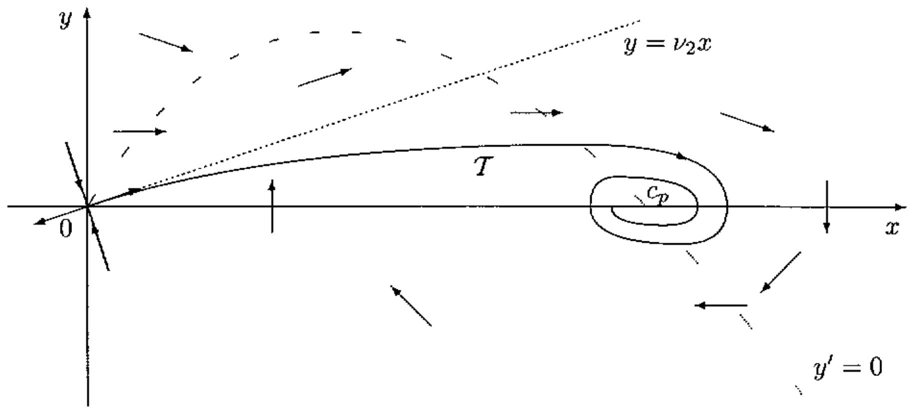

Figure 5: The flow generated by (9.9) for $p_S < p < p_{JL}$.

First consider the case $p > p_S$. Then $\beta > 0$ and the matrix A_1 has two real eigenvalues $\nu_{1,2} := -\frac{1}{2}(\beta \pm \sqrt{\beta^2 + 4\gamma})$ with $\nu_1 < 0 < \nu_2 = 2/(p-1)$. The corresponding eigenvectors (x_i, y_i) satisfy $y_i = \nu_i x_i$, $i = 1, 2$. The eigenvalues $\tilde{\nu}_{1,2} := -\frac{1}{2}(\beta \pm \sqrt{\beta^2 - 4\gamma(p-1)})$ of A_2 are real iff $\beta^2 \geq 4\gamma(p-1)$, that is iff $p \geq p_{JL}$.

Assume $p_S < p < p_{JL}$. In this case, the eigenvalues $\tilde{\nu}_1, \tilde{\nu}_2$ are complex and their real parts are negative so that the critical point $(c_p, 0)$ is a stable spiral. The flow for the planar system (9.9) is illustrated in Figure 5.

We are interested in the trajectory $\mathcal{T}$ emanating from the origin to the right half-space, since it represents the graph of any positive solution of (9.7) in the w-w' plane. This trajectory cannot hit the axis $x = 0$ again since the energy functional $\mathcal{E}$ is nonnegative on this axis, $\mathcal{E}(0,0) = 0$, $\beta > 0$ and (9.8) is true. Moreover, the corresponding solutions w exists for all $s \in \mathbb{R}$ and w, w' remain bounded for all $s \in \mathbb{R}$ due to (9.8). Consequently, $\mathcal{T}$ has to converge to the critical point $(c_p, 0)$ which corresponds to the singular solution $w_*(s) = r^{2/(p-1)}U_*(r) \equiv c_p$. Thus, if U_α is the unique local solution of (9.2) such that $U_\alpha(0) = \alpha > 0$, then its transform $w_\alpha(s) = r^{2/(p-1)}U_\alpha(r)$ exists globally and satisfies $w_\alpha(s) \to c_p$ as $s \to \infty$. Consequently, U_α exists globally and $r^{2/(p-1)}U_\alpha(r) \to c_p$ as $r \to \infty$. It is easily verified that the function $\tilde{U}_\alpha(r) := \alpha U_1(\alpha^{(p-1)/2}r)$ is a solution of (9.2) satisfying $\tilde{U}_\alpha(0) = \alpha$, hence $\tilde{U}_\alpha = U_\alpha$ by uniqueness. The graphs of w_α and w_1 in the w-w' plane are identical, so that there exists $s_\alpha \in \mathbb{R}$ such that $U_\alpha(e^s) = w_\alpha(s) = w_1(s - s_\alpha)$ for all $s \in \mathbb{R}$. Hence, given $\alpha_1 > \alpha_2 > 0$, $U_{\alpha_1}(r) = U_{\alpha_2}(r)$ for some $r > 0$ iff $w_1(s - s_{\alpha_1}) = w_1(s - s_{\alpha_2})$ for some $s \in \mathbb{R}$. This happens for infinitely many s since $\mathcal{T}$ spirals around the point $(c_p, 0)$. Similarly, $w_{\alpha_1}(s) = c_p$ for infinitely many s, hence U_{α_1} and U_* intersect infinitely many times.

Next consider the case $p \geq p_{JL}$. On the halfline $y = -\frac{\beta}{2}(x - c_p)$, $x < c_p$, we have for suitable $x_\theta \in (x, c_p)$:

$$\frac{y'}{x'} = -\beta - \frac{x}{y}(x^{p-1} - \gamma) = -\beta + \frac{2x(x^{p-1} - c_p^{p-1})}{\beta(x - c_p)}$$
$$= -\beta + \frac{2}{\beta}x(p-1)x_\theta^{p-2} < -\beta + \frac{2}{\beta}(p-1)\gamma \leq -\frac{\beta}{2}.$$

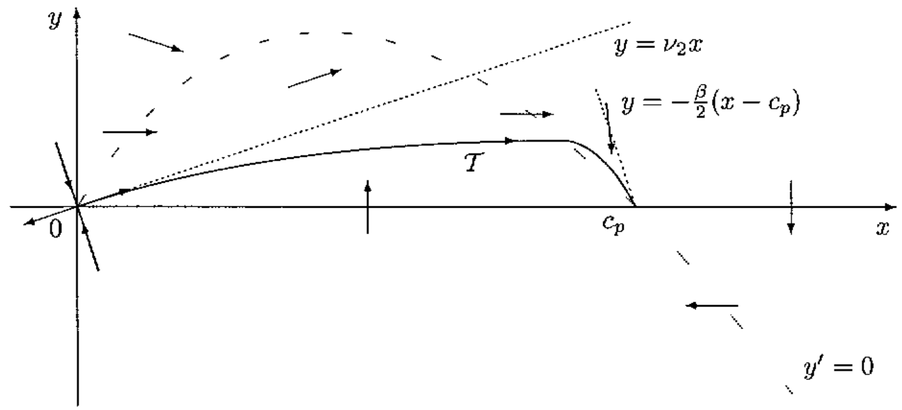

Figure 6: The flow generated by (9.9) for $p \geq p_{JL}$.

Consequently, the trajectory $\mathcal{T}$ ends up at $(c_p, 0)$ again but the x-coordinate is increasing along $\mathcal{T}$ (see Figure 6). Hence, the solutions U of (9.2) are ordered according to their values at $r = 0$, $U_* > U_{\alpha_1} > U_{\alpha_2}$ if $\alpha_1 > \alpha_2$.

Finally consider the case $p = p_S$. Then $\beta = 0$ and the energy functional $\mathcal{E}$ is constant along any solution. Since $\mathcal{E}(c_p, 0) < 0$ and $\mathcal{E}(0, y) > 0$ for $y \neq 0$, the trajectory $\mathcal{T}$ is a homoclinic orbit (see Figure 7).

Let w_α, s_α have the same meaning as above. Given $\alpha_1 \neq \alpha_2$, there exists a unique $s \in \mathbb{R}$ such that $w_1(s - s_{\alpha_1}) = w_1(s - s_{\alpha_2})$. Hence, the corresponding solutions $U_{\alpha_1}, U_{\alpha_2}$ of (9.2) intersect exactly once. Similarly, given $\alpha > 0$, we have $w_\alpha(s) = c_p$ for two values of s, so that U_α and U_* intersect twice. One can easily check that the function U_1 defined by (9.5) is a solution of (9.2) satisfying the initial condition $U_1(0) = 1$. □

Remarks 9.2. (i) The exponent p_{JL} appeared for the first time in [293] where the authors studied mainly problems with the nonlinearities $f(u) = \lambda(1 + au)^p$ and $f(u) = \lambda e^u$, $\lambda, a > 0$.

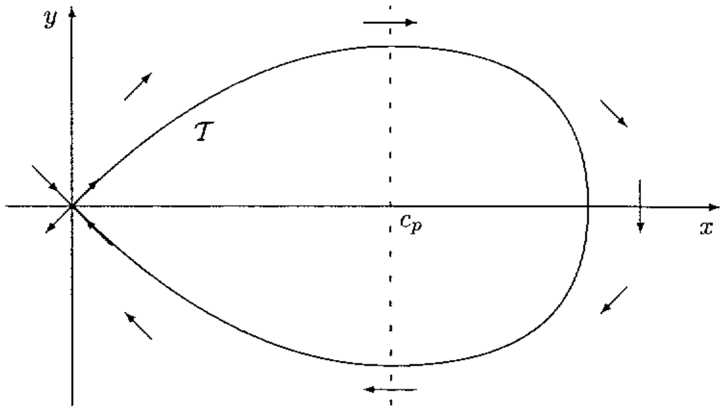

Figure 7: The flow generated by (9.9) for $p = p_S$.

(ii) The intersection properties of the solutions U in Theorem 9.1 play an important role in the study of stability and asymptotic behavior of solutions of the corresponding parabolic problem, see Sections 22, 23. □

Remark 9.3. Let $p = p_S$ and $a > 0$. For all $\alpha \geq M_0(a)$ with $M_0(a) > 0$ large enough, if V is a positive classical solution of

$$V'' + \frac{n-1}{r}V' + V^p = 0, \qquad 0 < r < a,$$

such that $V(a) = U_\alpha(a)$ and $\lim_{r\to 0} V(r) = \infty$, then V has to intersect U_α in $(0, a)$.

In fact, denoting $w_\alpha(s) := r^{2/(p-1)}U_\alpha(r)$, $s = \log r$, the rescaled function from the last proof, it suffices to chose $M_0(a)$ such that

$$w'_{M_0(a)}(\log a) < 0 \tag{9.10}$$

(hence $w'_\alpha(\log a) < 0$ for all $\alpha \geq M_0(a)$). Indeed the trajectory of $W(s) := r^{2/(p-1)}V(r)$, $s \in (-\infty, \log a)$, has to be a subset of a periodic orbit lying inside the trajectory $\mathcal{T}$ (see Figure 7). Due to (9.10) there exists $s_0 \in (-\infty, \log a)$ such that $w_\alpha(s_0) = W(s_0)$, hence $U_\alpha(e^{s_0}) = V(e^{s_0})$.

Note also that there exist infinitely many periodic orbits of (9.7) for $p = p_S$, corresponding to positive singular solutions of $u'' + \frac{n-1}{r}u' + u^p = 0$ for $r > 0$. □

Remark 9.4. Let $p > p_{JL}$. Since the trajectory $\mathcal{T}$ approaches the limit point $(c_p, 0)$ below the dotted line with slope $-\beta/2$ and $\tilde\nu_2 < -\beta/2 < \tilde\nu_1 < 0$, it has to converge along the eigenvector $(1, \tilde\nu_1)$ corresponding to the eigenvalue $\tilde\nu_1$, hence

$$\frac{y(s)}{x(s) - c_p} \to \tilde\nu_1 \quad \text{as} \quad s \to \infty.$$

Returning to the original variables and denoting $V(r) := U(r) - U_*(r)$ we obtain

$$\lim_{r\to\infty} \frac{rV'(r)}{V(r)} = \tilde\nu_1 - m, \tag{9.11}$$

where $m := 2/(p-1)$. Assuming that $V(r) = cr^{-\alpha} + h.o.t.$ for some $c \neq 0$ and $\alpha > m$, (9.11) guarantees $c < 0$ and $\alpha = m + \lambda_-$, where

$$\begin{aligned}\lambda_- := -\tilde\nu_1 &= \frac{1}{2}\big(\beta - \sqrt{\beta^2 - 4\gamma(p-1)}\big)\\ &= \frac{1}{2}\big(n - 2 - 2m - \sqrt{(n-2-2m)^2 - 8(n-2-m)}\big).\end{aligned}$$

This expansion is indeed true: In fact, a more precise asymptotic expansion of V was established in [260] and [334]. □

10. A priori bounds via the method of Hardy-Sobolev inequalities

A priori estimates of solutions can be used for the proof of existence and multiplicity results. Unlike the variational methods in sections 6 and 7, this approach does not require any variational structure of the problem and enables one to prove the existence of continuous branches of solutions.

Due to Theorem 7.8(ii) one cannot hope for a priori estimates of all solutions. The bifurcation diagrams in Figure 2 suggest that there is some hope for such estimates if we restrict ourselves to positive solutions and to the subcritical case.[3]

In the present and the following three sections we introduce four different methods which are often used in the proofs of a priori bounds for positive solutions of superlinear elliptic problems. We will study mainly the scalar problem

$$\left.\begin{aligned} -\Delta u &= f(x, u, \nabla u), && x \in \Omega,\\ u &= 0, && x \in \partial\Omega,\end{aligned}\right\} \tag{10.1}$$

where Ω is bounded and f is a sufficiently smooth function with superlinear growth in the u-variable. Some of the possible generalizations and modifications will be mentioned as remarks, others can be found in the subsequent chapters.

This section is devoted to the method of [99], which is based on a Hardy-type inequality and enables one to treat rather general nonlinearities f. On the

[3]In fact, in the subcritical case one can get a priori estimates of all solutions with bounded Morse indices (without the positivity assumption), see [49], [539], [32].

other hand, it requires an upper growth restriction corresponding to the limiting exponent

$$p_{BT} := \begin{cases} \infty & \text{if } n = 1, \\ (n+1)/(n-1) & \text{if } n > 1, \end{cases}$$

which is stronger than what is imposed by the methods in Sections 12 and 13 (for instance, in the particular case $f(x,u,\nabla u) = u^p$, we have to assume $p < p_{BT}$). However, the exponent p_{BT} is not technical and its role will be clarified in the next section.

Theorem 10.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded, $n \geq 3$, $\beta := p_{BT}$. Let $f : \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ be continuous and bounded on $\overline{\Omega} \times M \times \mathbb{R}^n$ for $M \subset \mathbb{R}_+$ bounded. Let*

$$\liminf_{u\to\infty} \frac{f(x,u,s)}{u} > \lambda_1, \quad \lim_{u\to\infty} \frac{f(x,u,s)}{u^\beta} = 0, \quad \textit{uniformly for } (x,s) \in \overline{\Omega} \times \mathbb{R}^n. \tag{10.2}$$

Then there exists $C > 0$ with the following property: If $t \geq 0$ and $u \in H_0^1 \cap L^\infty(\Omega)$ is a positive variational solution of

$$\left.\begin{aligned} -\Delta u &= f(x,u,\nabla u) + t\varphi_1, \qquad && x \in \Omega, \\ u &= 0, && x \in \partial\Omega, \end{aligned}\right\} \tag{10.3}$$

then

$$\|u\|_\infty + t \leq C. \tag{10.4}$$

Proof. We shall denote by C various positive constants which may vary from step to step but which are independent of u and t. Let $t \geq 0$ and u be a positive solution of (10.3). The proof of (10.4) will consist of the following three steps:

1. $\int_\Omega u\delta\,dx \leq C$, $t \leq C$ and $\int_\Omega f(x,u,\nabla u)\delta\,dx \leq C$,
2. $\|\nabla u\|_2 \leq C$,
3. $\|u\|_\infty \leq C$.

Step 1. Due to (10.2) there exist $C_1 > \lambda_1$ and $C_2 > 0$ such that $f(x,u,s) \geq C_1 u - C_2$ for all (x,u,s). Multiplying the equation in (10.3) by φ_1 yields

$$\begin{aligned} \lambda_1 \int_\Omega u\varphi_1\,dx &= \int_\Omega u(-\Delta\varphi_1)\,dx = \int_\Omega (-\Delta u)\varphi_1\,dx = \int_\Omega (f\varphi_1 + t\varphi_1^2)\,dx \\ &\geq C_1 \int_\Omega u\varphi_1\,dx - C_2 \int_\Omega \varphi_1\,dx + t\int_\Omega \varphi_1^2\,dx, \end{aligned} \tag{10.5}$$

where $f = f(x,u(x),\nabla u(x))$. This estimate can be written in the form

$$(C_1 - \lambda_1)\int_\Omega u\varphi_1\,dx + t\int_\Omega \varphi_1^2\,dx \leq C,$$

hence

$$\int_\Omega u\varphi_1\,dx \leq C \quad \text{and} \quad t \leq C. \tag{10.6}$$

Now (10.5) and $\delta \leq C\varphi_1$ guarantee

$$\int_\Omega f\delta\,dx \leq C\int_\Omega f\varphi_1\,dx = C\lambda_1 \int_\Omega u\varphi_1\,dx - Ct\int_\Omega \varphi_1^2\,dx \leq C. \tag{10.7}$$

Step 2. Multiplying the equation in (10.3) by u yields

$$\|\nabla u\|_2^2 = \int_\Omega |\nabla u|^2\,dx = \int_\Omega fu\,dx + t\int_\Omega \varphi_1 u\,dx \leq \int_\Omega fu\,dx + C. \tag{10.8}$$

Denoting $\alpha := 2/(n+1) \in (0,1)$ we have $\beta + 1/(1-\alpha) = 2/(1-\alpha)$. Given $\varepsilon > 0$ there exists $C_\varepsilon > 1$ such that

$$f(x,u,s) \leq \varepsilon u^\beta + C_\varepsilon. \tag{10.9}$$

Using Hölder's inequality, Step 1, (10.9) and Lemma 50.4 we obtain

$$\begin{aligned} \int_\Omega fu\,dx &= \int_\Omega (f^\alpha\delta^\alpha)\Big(f^{1-\alpha}\frac{u}{\delta^\alpha}\Big)\,dx \leq \Big(\int_\Omega f\delta\,dx\Big)^\alpha \Big(\int_\Omega f\frac{u^{1/(1-\alpha)}}{\delta^{\alpha/(1-\alpha)}}\,dx\Big)^{1-\alpha} \\ &\leq \varepsilon^{1-\alpha}\Big(\int_\Omega \frac{u^{\beta+1/(1-\alpha)}}{\delta^{\alpha/(1-\alpha)}}\,dx\Big)^{1-\alpha} + C_\varepsilon\Big(\int_\Omega \frac{u^{1/(1-\alpha)}}{\delta^{\alpha/(1-\alpha)}}\,dx\Big)^{1-\alpha} \\ &= \varepsilon^{1-\alpha}\Big\|\frac{u}{\delta^{\alpha/2}}\Big\|_{2/(1-\alpha)}^2 + C_\varepsilon\Big\|\frac{u}{\delta^\alpha}\Big\|_{1/(1-\alpha)} \leq \varepsilon^{1-\alpha}C\|\nabla u\|_2^2 + CC_\varepsilon\|\nabla u\|_2. \end{aligned}$$

This estimate and (10.8) guarantee

$$\|\nabla u\|_2 \leq C. \tag{10.10}$$

Step 3. Choose $p \in (n/2, n)$. Then

$$W^{2,p}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{and} \quad W^{1,2}(\Omega) \hookrightarrow L^{p(\beta-1)}(\Omega)$$

due to $n(\beta-1) < 2^*$. These embeddings, L^p-estimates (see Appendix A), (10.9), Step 1 and (10.10) imply

$$\begin{aligned} \|u\|_\infty &\leq C\|u\|_{2,p} \leq C\|f + t\varphi_1\|_p \leq \varepsilon\|u^\beta\|_p + C(C_\varepsilon + 1) \\ &\leq \varepsilon\|u\|_{p(\beta-1)}^{\beta-1}\|u\|_\infty + \tilde{C}_\varepsilon \leq \varepsilon\|\nabla u\|_2^{\beta-1}\|u\|_\infty + \tilde{C}_\varepsilon \leq \varepsilon C\|u\|_\infty + \tilde{C}_\varepsilon. \end{aligned}$$

Now choosing $\varepsilon > 0$ small enough yields $\|u\|_\infty < C$. □

Remarks 10.2. (i) The proof of Theorem 10.1 can be easily modified for more general second-order elliptic differential operators. In the case of a nonsymmetric operator one has to work with the first eigenfunction of the adjoint operator, of course. One could also allow more general nonlinearities (nonlocal, for example). The boundedness assumption on f could be relaxed as well.

(ii) The term $t\varphi_1$ in (10.3) is needed for the proof of existence of a positive solution of (10.3) with $t = 0$ (see Corollary 10.3 below). This lower order term does not play any significant role in a priori estimates in the following sections provided $t \le C$. Since this bound for t was proved in Step 1 of the proof of Theorem 10.1 by using only the lower bound for f in (10.2), in the following sections we shall restrict ourselves to the case $t = 0$ only.

(iii) A priori estimates of solutions of problems like (10.3) appeared first in [400] and [517]. The assumptions on the growth of f or the dimension n in these articles are more restrictive than those in Theorem 10.1 which is due to [99]. □

Corollary 10.3. *Let Ω and f be as in Theorem* 10.1 *and let*

$$\limsup_{u\to 0+} \frac{f(x,u,s)}{u} < \lambda_1 \qquad \text{uniformly for } (x,s) \in \overline{\Omega}\times\mathbb{R}^n. \tag{10.11}$$

Then problem (10.3) *with $t = 0$ possesses at least one positive solution u, with $u \in W^{2,q} \cap C_0(\Omega)$ for all finite q.*

Proof. Set $X := C^1(\overline{\Omega})$. Given $u \in X$ and $t \ge 0$, let $F_t(u) = w$ be the unique solution of the linear problem

$$\left.\begin{aligned} -\Delta w &= f(x,u,\nabla u) + t\varphi_1, && x \in \Omega,\\ w &= 0, && x \in \partial\Omega \end{aligned}\right\} \tag{10.12}$$

(cf. Theorem 47.3(i)). Note that, since $f(\cdot,u,\nabla u) \in L^\infty(\Omega)$, we have $u \in W^{2,q} \cap C_0(\Omega)$ for all finite q. Then $F_t : X \to X$ is compact and we are looking for a positive fixed point of F_0.

Let $\|u\|_X = r \ll 1$, $\tau \in [0,1]$ and assume $\tau F_0(u) = u$. Multiplying the equation in (10.12) by u and applying (10.11) yield

$$\int_\Omega |\nabla u|^2\,dx = \tau\int_\Omega fu\,dx \le (\lambda_1 - \varepsilon)\int_\Omega u^2\,dx,$$

which contradicts (1.3). Hence $\tau F_0(u) \ne u$ and the homotopy invariance of the topological degree implies

$$\deg(I - F_0, 0, B_r) = \deg(I, 0, B_r) = 1, \tag{10.13}$$

where I denotes the identity and $B_r := \{u \in X : \|u\|_X < r\}$.

Let $\|u\|_X = R$. If R is large enough, then Theorem 10.1 and L^p-estimates (see Appendix A) imply $F_t(u) \ne u$ for any $t \ge 0$. The same theorem implies also $F_T(u) \ne u$ provided T is large enough. Consequently,

$$\deg(I - F_0, 0, B_R) = \deg(I - F_T, 0, B_R) = 0. \tag{10.14}$$

Now (10.13) and (10.14) guarantee $\deg(I - F_0, 0, B_R \setminus \bar{B}_r) = -1$, hence there exists $u \in B_R \setminus \bar{B}_r$ such that $F_0(u) = u$. The positivity of u is a consequence of the maximum principle. □

In what follows we present an alternative proof of Theorem 10.1 in the special case $f(x,u,s) = |u|^{p-1}u$, $1 < p < p_{BT}$, $n \ge 1$. Instead of Hardy's inequality we shall use the following lemma (see [89], [450], and cf. also [143] and the references in [450, Remark 4.1]). It provides a useful singular test-function and will also be used later in Section 26.

Lemma 10.4. *Assume Ω bounded and $0 < \alpha < 1$. Then the problem*

$$\left.\begin{aligned} -\Delta\xi &= \varphi_1^{-\alpha}, && x \in \Omega,\\ \xi &= 0, && x \in \partial\Omega \end{aligned}\right\} \tag{10.15}$$

admits a unique classical solution $\xi \in C(\overline{\Omega}) \cap C^2(\Omega)$. Moreover, we have $\varphi_1^{-\alpha} \in L^1(\Omega)$, $\xi \in H_0^1(\Omega)$, and

$$\xi(x) \le C(\Omega,\alpha)\delta(x), \quad x \in \Omega. \tag{10.16}$$

Proof. Define $h(s) = 3s - s^{2-\alpha}$, $s \ge 0$. The function $h \in C^1([0,\infty)) \cap C^2((0,\infty))$ satisfies

$$h' = 3 - (2-\alpha)s^{1-\alpha}, \quad -h'' = (2-\alpha)(1-\alpha)s^{-\alpha}, \quad s > 0$$

and

$$h(s) \le 3s, \quad h'(s) \ge 1, \quad \text{for all } s \in [0,1].$$

Let $\varphi = \|\varphi_1\|_\infty^{-1}\varphi_1$, and set $v(x) = h(\varphi(x))$. Simple computation yields

$$\begin{aligned} -\Delta v &= -h''(\varphi)|\nabla\varphi|^2 - h'(\varphi)\Delta\varphi\\ &= C_1\varphi^{-\alpha}|\nabla\varphi|^2 + \lambda_1 h'(\varphi)\varphi\\ &\ge C_1\varphi^{-\alpha}|\nabla\varphi|^2 + \lambda_1\varphi. \end{aligned}$$

Now, for $\delta(x) \le \varepsilon$ small enough, we have $|\nabla\varphi|^2 \ge \eta > 0$, hence $-\Delta v \ge C_1\eta\varphi^{-\alpha}$. On the other hand, for $\delta(x) \ge \varepsilon$, we have $\varphi \ge c > 0$, hence $-\Delta v \ge \lambda_1 c \ge C_2\varphi^{-\alpha}$. We conclude that for some $c > 0$, $w := cv$ satisfies

$$-\Delta w \ge \varphi_1^{-\alpha} \quad \text{and} \quad w(x) \le C_3\delta(x), \quad \text{for all } x \in \Omega. \tag{10.17}$$

Next, for all $\varepsilon > 0$, let ξ_ε be the (classical) solution of $-\Delta\xi_\varepsilon = (\varphi_1+\varepsilon)^{-\alpha}$ in Ω, with $\xi_\varepsilon = 0$ on $\partial\Omega$. By (10.17) and the maximum principle, we have

$$\xi_\varepsilon(x) \le w(x) \le C_3\delta(x) \le C_4, \quad x \in \Omega \tag{10.18}$$

and ξ_ε is increasing as ε decreases to 0. Denote by ξ the (pointwise) limit of ξ_ε. Elliptic estimates along with (10.18) imply that $\xi \in C(\overline{\Omega})\cap C^2(\Omega)$, that ξ satisfies (10.16) and is a classical solution of (10.15). The uniqueness follows immediately from the maximum principle.

The fact that $\varphi_1^{-\alpha} \in L^1(\Omega)$ can be easily deduced from the inequality $\varphi_1 \ge c\delta$, by flattening the boundary and using a partition of unity (see e.g. [485] for details). Finally, to show that $\xi \in H_0^1(\Omega)$, it suffices to note that, since $\alpha < 1$,

$$\int_\Omega |\nabla\xi_\varepsilon|^2 = -\int_\Omega \xi_\varepsilon\Delta\xi_\varepsilon = \int_\Omega \xi_\varepsilon(\varphi_1+\varepsilon)^{-\alpha} \le C_4\int_\Omega \varphi_1^{-\alpha} < \infty. \quad \square$$

Alternative proof of Theorem 10.1 for $f = u^p$, $t = 0$. Let $\varepsilon > 0$ be small and $\alpha := r'/r$, where r is defined by $1/r = 1/2 - \varepsilon/(p-1)$. Let ξ be the solution of (10.15). As in Step 1 of the proof of Theorem 10.1 we obtain $\int_\Omega u^p\delta\,dx \le C$. Testing the equation with ξ, we obtain

$$\int_\Omega u\varphi_1^{-\alpha}\,dx = \int_\Omega \nabla u\cdot\nabla\xi\,dx = \int_\Omega (-\Delta u)\xi\,dx = \int_\Omega u^p\xi\,dx \le C$$

(where we used $\varphi_1^{-\alpha} \in L^1(\Omega)$ and $\xi \in H_0^1(\Omega)$). Denoting $p_\varepsilon := (p+1)/2 - \varepsilon$, we get

$$\begin{aligned}\int_\Omega u^{p_\varepsilon}\,dx &= \int_\Omega \bigl(u^{p/r}\varphi_1^{1/r}\bigr)\bigl(u^{1/r'}\varphi_1^{-1/r}\bigr)\,dx \\ &\le \Bigl(\int_\Omega u^p\varphi_1\,dx\Bigr)^{1/r}\Bigl(\int_\Omega u\varphi_1^{-\alpha}\,dx\Bigr)^{1/r'} \le C.\end{aligned}$$

Define $\theta \in (0, p+1)$ by $\theta/p_\varepsilon + (p+1-\theta)/2^* = 1$. Then $p+1-\theta < 2$ provided ε is small enough and the interpolation inequality yields

$$\int_\Omega |\nabla u|^2\,dx = \int_\Omega u^{p+1}\,dx = \|u\|_{p+1}^{p+1} \le \|u\|_{p_\varepsilon}^\theta\|u\|_{2^*}^{p+1-\theta} \le C\|\nabla u\|_2^{p+1-\theta},$$

which guarantees a bound for u in $W^{1,2}(\Omega)$. The rest of the proof is the same as in the proof of Theorem 10.1 (Step 3). $\square$

11. A priori bounds via bootstrap in L^p_δ-spaces

This section is devoted to the L^p_δ bootstrap method, which, in the scalar case, was developed independently in [85], [449]. It applies to problem (10.1) under essentially the same assumptions on the nonlinearities f as in the method of the previous section, with a growth restriction still given by the exponent p_{BT} of Section 10. However, unlike that method (and those in the next two sections), it applies to very weak solutions. The optimality of the L^p_δ bootstrap method was studied in [489] and it turns out that the exponent p_{BT} is optimal for the regularity of very weak solutions, thus showing the critical role played by this exponent for problems of the form (10.1).

Let us point out that in the case of systems, studied in [449], the growth restrictions of the L^p_δ bootstrap method become much weaker than those imposed by the (generalization of the) method of Hardy-Sobolev inequalities (see Section 31).

In this section, by a solution u of (10.1), we understand a very weak (or L^1_δ-) solution, cf. Definition 3.1. Namely, if f does not depend on ∇u, this means that

$$u \in L^1(\Omega), \quad f(\cdot,u) \in L^1_\delta(\Omega), \tag{11.1}$$

and

$$-\int_\Omega u\Delta\varphi = \int_\Omega f(\cdot,u)\varphi, \quad \text{for all } \varphi \in C^2(\overline{\Omega}),\ \varphi_{|\partial\Omega} = 0. \tag{11.2}$$

If f depends on ∇u, we assume in addition that ∇u is a function, i.e. $\nabla u \in L^1_{\rm loc}(\Omega)$ and we replace $f(\cdot,u)$ by $f(\cdot,u,\nabla u)$ in (11.1)–(11.2).

Remark 11.1. If $u \in L^1(\Omega)$ and $\Delta u \in L^1_\delta(\Omega)$ (where Δu is understood in the distribution sense), we say that $u = 0$ on $\partial\Omega$ in the weak sense if

$$\int_\Omega u\,\Delta\varphi = \int_\Omega \varphi\,\Delta u \quad \text{for all } \varphi \in C^2(\overline{\Omega}),\ \varphi_{|\partial\Omega} = 0.$$

If (11.1) is satisfied (and $\nabla u \in L^1_{\rm loc}(\Omega)$ in case f depends on ∇u), then u is a very weak solution of (10.1) if and only if it solves the differential equations in (10.1) in the distribution sense and the boundary conditions in the weak sense. $\square$

Theorem 11.2. *Assume Ω bounded and $1 < p < p_{BT}$. Let $f : \overline{\Omega}\times\mathbb{R}_+\times\mathbb{R}^n \to \mathbb{R}_+$ be continuous. Assume*

$$f(x,u,s) \le C_1(1+u^p), \quad x \in \Omega, \quad u \ge 0, \quad s \in \mathbb{R}^n \tag{11.3}$$

and

$$f(x,u,s) \ge \lambda u - C_1, \quad x \in \Omega, \quad u \ge 0, \quad s \in \mathbb{R}^n \quad \textit{for some } \lambda > \lambda_1. \tag{11.4}$$

There exists $C > 0$ such that if u is a nonnegative very weak solution of (10.1), *then $u \in L^\infty(\Omega)$ and*

$$\|u\|_\infty \le C.$$

Condition (11.4) can be weakened or replaced by other conditions of different form. For instance, by applying the same method, we obtain regularity and a priori estimates for the following simple equation:

$$\left.\begin{aligned} -\Delta u &= a(x)u^p, & x &\in \Omega,\\ u &= 0, & x &\in \partial\Omega. \end{aligned}\right\} \tag{11.5}$$

Theorem 11.3. *Assume Ω bounded and $a \in L^\infty(\Omega)$, $a \ge 0$, $a \not\equiv 0$ and $1 < p < p_{BT}$. Then the conclusions of Theorem* 11.2 *remain valid for problem* (11.5).

Remarks 11.4. (i) The growth condition (11.3) in Theorem 11.2 is slightly stronger than that in Theorem 10.1 (where (10.2) allows some "almost critical" f's).

(ii) Under the assumptions of Theorems 11.2 and 11.3, as a consequence of standard regularity results for linear elliptic equations, we moreover obtain $u \in C_0 \cap W^{2,q}(\Omega)$ for all finite q (argue similarly as in the proof of Corollary 3.4, using the uniqueness part of Theorem 49.1 instead of Proposition 52.3). □

The optimality of the exponent p_{BT} in Theorems 11.2 and 11.3 is shown by the following result from [489].

Theorem 11.5. *Assume Ω bounded and $p > p_{BT}$. Then there exists a function $a \in L^\infty(\Omega)$, $a \ge 0$, $a \not\equiv 0$, such that problem* (11.5) *admits a positive very weak solution u such that*

$$u \notin L^\infty(\Omega).$$

The method of proof of Theorems 11.2–11.3 is based on bootstrap and uses the L^p_δ regularity theory of the Laplacian (cf. Theorem 49.2 and Proposition 49.5 in Appendix C).

Proof of Theorem 11.2. *Step 1.* Initialization. By (10.6), (10.7) in the proof of Theorem 10.1, we know that

$$\|u\|_{1,\delta} \le C, \qquad \|f(\cdot, u, \nabla u)\|_{1,\delta} \le C. \tag{11.6}$$

Since $p < p_{BT}$, we may fix $\rho > 1$ and k_0 such that

$$\max\Big(p, \frac{n+1}{2}\Big(p - \frac{1}{\rho}\Big)\Big) < k_0 < \frac{n+1}{n-1}.$$

By (11.6) and Proposition 49.5, it follows that $\|u\|_{k_0,\delta} \le C$.

Step 2. Bootstrap. Put $k_i = k_0\rho^i$, $i = 1, 2, \dots$. Assume that there holds

$$\|u\|_{k_i,\delta} \le C(i) \tag{11.7}$$

for some $i \ge 0$ (this is true for $i = 0$ by Step 1). Since

$$\frac{p}{k_i} - \frac{1}{k_{i+1}} = \frac{1}{k_0\rho^i}\Big(p - \frac{1}{\rho}\Big) < \frac{2}{n+1},$$

by using Theorem 49.2(i) and (11.3), we obtain

$$\begin{aligned} \|u\|_{k_{i+1},\delta} &\le C\|\Delta u\|_{k_i/p,\delta} = C\|f\|_{k_i/p,\delta}\\ &\le C(1 + \|v^p\|_{k_i/p,\delta}) = C(1 + \|v\|^p_{k_i,\delta}) \le C. \end{aligned}$$

By induction, it follows that (11.7) is true for all integers i. Taking i large enough, we thus have (11.7) for some $k_i > (n+1)p/2$. Applying Theorem 49.2(i) and (11.3) once more, and Remark 1.1, we obtain $\|u\|_\infty \le C$. □

Proof of Theorem 11.3. We only need to modify Step 1, the bootstrap step being then unchanged.

Assume that u is a nonnegative (very weak) solution of (11.5). It follows from the quantitative version of Hopf's lemma (see Remark 49.12(i) in Appendix C) that

$$u \ge c\Big(\int_\Omega au^p\delta\,dy\Big)\,\delta \ge c_1\Big(\int_\Omega au^p\varphi_1\,dy\Big)\,\varphi_1,$$

for some constant $c_1 > 0$ depending only on Ω. We deduce that

$$\int_\Omega au^p\varphi_1\,dx \ge c_1^p\Big(\int_\Omega au^p\varphi_1\,dx\Big)^p \int_\Omega a\varphi_1^{p+1}\,dx \ge 2\int_\Omega au^p\varphi_1\,dx - C,$$

hence

$$\lambda_1 \int_\Omega u\varphi_1\,dx = \int_\Omega au^p\varphi_1\,dx \le C. \quad \square$$

We now turn to the proof of Theorem 11.5. It is based on Lemma 49.13 from Appendix C, where a singular solution of the linear Laplace equation with an appropriate right-hand side belonging to L^1_δ is constructed. The right-hand side has to possess suitable boundary singularities, supported in a conical subdomain of Ω. In order to re-construct a posteriori the coefficient $a(x)$, the key point is the lower estimate (11.8) for the solution in the same cone.

Proof of Theorem 11.5. Assume that $0 \in \partial\Omega$ without loss of generality. Let $\alpha = 2/(p-1)$. By assumption, we have $\alpha < n - 1$. By Lemma 49.13, there exist $R > 0$ and a revolution cone Σ_1 of vertex 0, with $\Sigma := \Sigma_1 \cap B_{2R} \subset \Omega$, such that the function

$$\phi := |x|^{-(\alpha+2)}\chi_\Sigma$$

belongs to L^1_δ and such that the (very weak) solution $u > 0$ of

$$\left.\begin{aligned} -\Delta u &= \phi, && x \in \Omega, \\ u &= 0, && x \in \partial\Omega \end{aligned}\right\}$$

satisfies

$$u \geq C|x|^{-\alpha}\chi_\Sigma. \tag{11.8}$$

Therefore, we have $u \notin L^\infty$ and

$$u^p \geq C|x|^{-\alpha p}\chi_\Sigma = C|x|^{-(\alpha+2)}\chi_\Sigma = C\phi.$$

Setting $a(x) = \phi/u^p \geq 0$, we get $-\Delta u = \phi = a(x)u^p$ and $a(x) \leq 1/C$, hence $a \in L^\infty$. The proof is complete. □

Remarks 11.6. Localization of singularities. (a) In Theorem 11.5, it is to be noted that, in spite of the imposed homogeneous Dirichlet boundary condition, the singularity of the solution occurs at a boundary point, actually a single point. The boundary conditions continue to be satisfied not only in the weak sense but also in the sense of traces (see Remark 49.4(c) in Appendix C).

(b) If we assume that $p < p_{sg}$ and that a given weak solution of (11.5) is bounded near the boundary, then one can use usual Lebesgue spaces instead of L^p_δ-spaces in the proof of Theorem 11.2, to show that the solution is bounded in Ω. Therefore, the occurrence of boundary singularities is necessary if $p_{BT} < p < p_{sg}$. On the other hand, when $p > p_{sg}$, the situation is different and much easier, since it is then not difficult to construct examples of similar equations with only an interior singularity (see Remarks 3.6).

(c) The support of a in Theorem 11.5 can be localized in an arbitrarily small neighborhood of a boundary point. However, it is also possible to construct an example where the function a is positive in Ω, uniformly away from $\partial\Omega$ (see [489] for details). □

Remarks 11.7. (a) **The cases $f(u) = u^p$ and $p = p_{BT}$.** Similar counterexamples as in Theorem 11.5 have been constructed recently in [155] for the model problem (3.10) ($a(x) \equiv 1$) when $p > p_{BT}$ is close to p_{BT}. Moreover the critical case $p = p_{BT}$ was shown to belong to the singular case. Related results have also been announced in [81].

(b) **Variable critical exponents in nonsmooth domains.** The notion of very weak solution has been recently extended in [362] to the case of some nonsmooth domains, namely Lipschitz domains, and generalizations of Theorems 11.2 and 11.5 have been obtained. For suitable cone-shaped domains, the analogue of the exponent p_{BT} was computed. Interestingly, it was found to depend on the domain and to be smaller than $(n+1)/(n-1)$. □

12. A priori bounds via the rescaling method

In this section we present a priori estimates of solutions of (10.3) based on rescaling and Liouville-type theorems. In this context, this method was first used in [241]. In comparison to the method of Section 10, it requires a rather precise asymptotic behavior for f as $u \to \infty$ (f has to behave like u^p for u large) but the growth condition on f is optimal ($p < p_S$). The method also works for general second-order elliptic operators but for simplicity we restrict ourselves to the Laplace operator. As explained in Remark 10.2(ii) we consider the case $t = 0$ only.

Theorem 12.1. *Assume Ω bounded, $1 < p < p_S$, $a \in C(\overline{\Omega})$, $a(x) \geq a_0 > 0$ for all $x \in \overline{\Omega}$, $g \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, and*

$$|g(x,u,s)| \leq C\big(1 + |u|^q + |s|^r\big), \qquad \text{where } q < p,\ r < \frac{2p}{p+1}. \tag{12.1}$$

Then there exists $C > 0$ such that any positive strong solution $u \in C^1(\overline{\Omega})$ of

$$\left.\begin{aligned} -\Delta u &= a(x)u^p + g(x,u,\nabla u), && x \in \Omega, \\ u &= 0, && x \in \partial\Omega \end{aligned}\right\} \tag{12.2}$$

satisfies $\|u\|_\infty \leq C$.

Remark 12.2. Here, u being a strong solution means that $u \in W^{2,1}_{loc}(\Omega)$ and u satisfies the differential equation a.e. in Ω. Since we also assume $u \in C^1(\overline{\Omega})$, Remarks 47.4(i) and (iii), actually imply $u \in W^{2,q}(\Omega)$ for all finite q. □

Proof of Theorem 12.1. Assume the contrary. Then there exist positive solutions u_j of (12.2) such that $\|u_j\|_\infty \to \infty$ as $j \to \infty$. Let $x_j \in \Omega$ be such that

$$u_j(x_j) + |\nabla u_j(x_j)|^{2/(p+1)} = \sup_\Omega\big(u_j + |\nabla u_j|^{2/(p+1)}\big) =: M_j$$

and let $d_j := \operatorname{dist}(x_j, \partial\Omega)$. Since $\overline{\Omega}$ is compact, we may assume $x_j \to x_0$ for some $x_0 \in \overline{\Omega}$. Set $\kappa_j := M_j^{-(p-1)/2}$. The sequence d_j/κ_j is either unbounded or bounded. In the former case we may assume $d_j/\kappa_j \to \infty$, in the latter $d_j/\kappa_j \to c \geq 0$.

Case 1. Let $d_j/\kappa_j \to \infty$. Set

$$v_j(y) := \frac{1}{M_j}u_j(x), \qquad y := \frac{x - x_j}{\kappa_j},$$

and $\Omega_j := \{y \in \mathbb{R}^n : |y| < d_j/\kappa_j\}$. Then

$$v_j + |\nabla v_j|^{2/(p+1)} \leq v_j(0) + |\nabla v_j(0)|^{2/(p+1)} = 1 \tag{12.3}$$

and

$$-\Delta v_j(y) = a(\kappa_j y + x_j)v_j^p(y) + g_j(y), \qquad y \in \Omega_j, \tag{12.4}$$

where

$$g_j(y) := \kappa_j^{2p/(p-1)} g\big(\kappa_j y + x_j, \kappa_j^{-2/(p-1)} v_j(y), \kappa_j^{(p+1)/(p-1)} \nabla v_j(y)\big)$$

satisfies

$$|g_j| \le C\kappa_j^{\varepsilon}, \qquad \varepsilon := \min\big(2(p-q), 2p-(p+1)r\big)/(p-1). \tag{12.5}$$

Interior elliptic L^p-estimates (see Appendix A) guarantee that v_j are locally bounded in $W^{2,z}$ for any $z > 1$ (uniformly with respect to j). Let $\alpha \in (0,1)$, $R > 0$ and $B_R := \{y \in \mathbb{R}^n : |y| < R\}$. There exists $z = z(\alpha) > 1$ such that $W^{2,z}(B_R)$ is compactly embedded into $BUC^{1+\alpha}(B_R)$. Consequently, we may assume $v_j \to v$ in $C^{1+\alpha}$. Passing to the limit in (12.4) and (12.3) we see that v is a positive (classical) solution of

$$-\Delta v = a(x_0)v^p \quad \text{in } \mathbb{R}^n,$$

which contradicts Theorem 8.1.

Case 2. Let $d_j/\kappa_j \to c \ge 0$. Let $\tilde{x}_j \in \partial\Omega$ be such that $d_j = |x_j - \tilde{x}_j|$. For any j we can choose a local coordinate $z = z_{(j)} = (z^1, z^2, \dots, z^n)$ in an ε-neighborhood U_j of $\tilde{x}_j$ such that the image of the boundary $\partial\Omega$ will be contained in the hyperplane $z^1 = 0$, $\tilde{x}_j$ becomes 0, x_j becomes $z_j := (d_j, 0, 0, \dots, 0)$, and the image of U_j will contain the set $\{z : |z| < \varepsilon'\}$ for some $\varepsilon' > 0$. We may assume that $\varepsilon, \varepsilon'$ are independent of j and the local charts are uniformly bounded in C^2. In these new coordinates, the equation for $w = w_j(z) = u_j(x)$ becomes

$$\left.\begin{aligned} -\sum_{i,k} a^{ik}(z)\frac{\partial^2 w}{\partial z^i \partial z^k} - \sum_i b^i(z)\frac{\partial w}{\partial z^i} &= a(x(z))w^p + \tilde{g}(z), && |z| < \varepsilon,\ z^1 > 0, \\ w &= 0, && |z| < \varepsilon,\ z^1 = 0, \end{aligned}\right\} \tag{12.6}$$

where $\tilde{g}(z) := g\big(x(z), w(z), D(z)\nabla_z w(z)\big)$, $D = D_{(j)} = (\partial z^i/\partial x^k)_{i,k}$, $b^i = b^i_{(j)} = \Delta z^i$, $a^{ik} = a^{ik}_{(j)} = \sum_\ell \frac{\partial z^i}{\partial x^\ell}\frac{\partial z^k}{\partial x^\ell}$, hence $A = A_{(j)} := (a^{ik}_{(j)})_{i,k} = D \cdot {}^tD$, and the $A_{(j)}$ are uniformly elliptic. Also, since $\partial\Omega$ is uniformly C^2, it follows that the $a^{ik}_{(j)}$ are uniformly bounded in C^1 and the $b^i_{(j)}$ in L^∞. Moreover, since $D(0)$ is a Euclidean transformation, it follows that $A_{(j)}(0) = D(0) \cdot {}^tD(0) = \mathrm{Id}$. Set

$$v_j(y, s) := \frac{1}{M_j} w_j(\kappa_j y + z_j),$$

where

$$y \in \Omega_j := \Big\{ y : \Big| y - \frac{z_j}{\kappa_j}\Big| < \frac{\varepsilon'}{\kappa_j},\ y^1 > -\frac{d_j}{\kappa_j} \Big\}.$$

Then v_j is a solution of

$$\begin{aligned} -\sum_{i,k} a^{ik}(\kappa_j y + z_j)\frac{\partial^2 v}{\partial y^i \partial y^k} - \kappa_j \sum_i b^i(\kappa_j y + z_j)\frac{\partial v}{\partial y^i} & \\ = a(x(\kappa_j y + z_j))v^p + g_j & \qquad \text{in } \Omega_j, \\ v = 0 & \qquad \text{on } \{y \in \partial\Omega_j : y^1 = -d_j/\kappa_j\}, \end{aligned}$$

where

$$g_j(y) := \kappa_j^{2p/(p-1)} g\big(x(\kappa_j y + z_j), \kappa_j^{-2/(p-1)} v(y), \kappa_j^{-(p+1)/(p-1)} D(\kappa_j y + z_j)\nabla v(y)\big)$$

satisfies (12.5). Interior-boundary L^p-estimates (see Appendix A) and the bounds on the coefficients $a^{ik}_{(j)}$, $b^i_{(j)}$ again yield a subsequence of $\{v_j\}$ converging to a positive (classical) solution v of

$$\begin{aligned} \Delta v &= a(x_0)v^p, && y_1 > -c, \\ v &= 0, && y_1 = -c, \end{aligned}$$

which contradicts Theorem 8.2. □

Remarks 12.3. (i) If g is independent of the gradient variable, then it is sufficient to choose $M_k := \sup u_k$ in the proof of Theorem 12.1.

(ii) **Indefinite coefficients.** Assume that the function a in problem (12.2) changes sign. Under suitable assumptions on a, g and p one can still use the method of [241] in order to get a priori bounds for positive solutions (see [74], [19] and [167], for example). In addition to the limiting problems in the proof of Theorem 12.1 one has to deal with problems of the form

$$-\Delta u = h(y)u^p, \qquad y \in \mathbb{R}^n,$$

where typically $h(y) = |y_1|^\alpha y_1$ for some $\alpha \ge 0$. In some cases, a combination of the above approach with other arguments (moving planes, energy, ...) yields the a priori bounds, see [124], [453], [237] and the references therein. Of course, if the problem has variational structure, then the existence of nontrivial solutions can often be proved by variational or dynamical methods, see [8], [75], [7], [257], [121], [3] and the references therein.

(iii) The rescaling method is sometimes referred to as the "blow-up method", because one performs a zoom of the microscopic scales of the solution. Here we shall not use this terminology, in order to avoid confusion with the blow-up phenomenon in the parabolic problem. □

13. A priori bounds via moving planes and Pohozaev's identity

In this section we describe the method of a priori estimates of solutions of (2.1) due to [183]. Similarly as in the preceding section, the growth condition for function f will be optimal. The advantage of this method consists in the fact that it does neither require precise asymptotic behavior of f for u large nor Liouville-type theorems. On the other hand, the symmetry of the Laplace operator plays an important role, f cannot depend on ∇u in a general way and we also have to assume that either Ω is convex or f satisfies a restrictive monotonicity condition, see (13.3) below. The assumptions for a general function $f = f(x,u)$ are rather complicated (see [183, Remark 1.5]) and therefore we restrict ourselves to the case $f = f(u)$. Hence, we shall study positive solutions of the problem

$$\left.\begin{aligned} -\Delta u &= f(u), & x &\in \Omega, \\ u &= 0, & x &\in \partial\Omega. \end{aligned}\right\} \tag{13.1}$$

Theorem 13.1. *Assume $n \geq 2$ and Ω bounded. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be locally Lipschitz continuous and assume*

$$\liminf_{u\to\infty} \frac{f(u)}{u} > \lambda_1, \qquad \lim_{u\to\infty} \frac{f(u)}{u^\sigma} = 0,$$

where $\sigma = p_S$ if $n \geq 3$, $\sigma < \infty$ is arbitrary if $n = 2$. Let one of the following assumptions be satisfied:

(i) *Ω is convex and*

$$\limsup_{u\to\infty} \frac{uf(u) - \theta F(u)}{u^2 f(u)^\kappa} \leq 0, \qquad \theta \in [0, 2^*), \tag{13.2}$$

where $\kappa = 2/n$.

(ii) *Condition (13.2) is satisfied with $\kappa = 2/n$ and, in the case $n \geq 3$,*

$$\text{the function } u \mapsto f(u)u^{-p_S} \text{ is nonincreasing on } (0,\infty). \tag{13.3}$$

(iii) *Condition (13.2) is satisfied with $\kappa = 2/(n+1)$, $n \geq 3$, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1, Γ_2 are closed and satisfy*

  (1) *at every point of Γ_1, all sectional curvatures of Γ_1 are bounded away from 0 by a positive constant a;*

  (2) *there exists $x_0 \in \mathbb{R}^n$ such that $(x - x_0, \nu(x)) \leq 0$ for all $x \in \Gamma_2$.*

Then there exists $C > 0$ such that $\|u\|_\infty < C$ for any positive classical solution u of (13.1).

In view of the proof we set some notation. For each $\varepsilon > 0$, let

$$\Omega_\varepsilon := \{z \in \Omega : \delta(z) < \varepsilon\}.$$

For $y \in \partial\Omega$ and $\lambda > 0$, we define

$$T(y,\lambda) := \{x \in \mathbb{R}^n : (y - x, \nu(y)) = \lambda\},$$
$$\Sigma(y,\lambda) := \{x \in \Omega : (y - x, \nu(y)) \leq \lambda\},$$

we denote by $R(y,\lambda)$ the reflection with respect to the hyperplane $T(y,\lambda)$ and we set $\Sigma'(y,\lambda) := R(y,\lambda)\Sigma(y,\lambda)$. We need the following lemma.

Lemma 13.2. *Assume Ω bounded and convex, $\lambda_0 > 0$, and $0 \leq u \in C(\overline{\Omega}) \cap C^1(\Omega)$. Assume that*

$$(\nabla u(x), \nu(y)) \leq 0, \qquad y \in \partial\Omega,\ x \in \Sigma(y,\lambda_0). \tag{13.4}$$

Then

$$\sup_{\Omega_\varepsilon} u \leq C \int_\Omega u\varphi_1\, dx,$$

where $\varepsilon, C > 0$ depend only on Ω and λ_0.

Proof. Let us first recall that

$$\nu(\partial\Omega) = S^{n-1}. \tag{13.5}$$

This follows from a standard degree argument. We give the proof for completeness. Assume without loss of generality that $0 \in \Omega$ and select $\tilde\nu$, a continuous extension of ν to $\overline{\Omega}$. The homotopy $H_1(t,x) := t\tilde\nu(x) + (1-t)x$ has no zero on $\partial\Omega$, since $(x, \nu(x)) \geq 0$ on $\partial\Omega$ due to the convexity of Ω. Therefore $d(\tilde\nu, 0, \Omega) = d(id, 0, \Omega) = 1$, where d denotes the Brouwer degree. Assume for contradiction that $\eta \notin \nu(\partial\Omega)$ for some $\eta \in S^{n-1}$. Then the homotopy $H_2(t,x) = t\tilde\nu(x) - (1-t)\eta$ has no zero on $\partial\Omega$. Consequently $d(\tilde\nu, 0, \Omega) = d(-\eta, 0, \Omega) = 0$, a contradiction which proves (13.5).

Next, by decreasing λ_0 if necessary, we may assume that

$$\{y - \lambda\nu(y) \in \mathbb{R}^n : \lambda \in (0, \lambda_0]\} \subset \Omega, \qquad y \in \partial\Omega. \tag{13.6}$$

Let $\varepsilon \in (0, \lambda_0/4]$, $x \in \Omega_\varepsilon$, and let $\tilde x \in \partial\Omega$ satisfy $|x - \tilde x| = \delta(x)$. Notice that $\tilde x$ is uniquely determined and $(\tilde x - x)/|\tilde x - x| = \nu(\tilde x)$ if ε is small. Let $\alpha \in (0,1)$ and let $\eta \in S^{n-1}$ be such that $(\eta, \nu(\tilde x)) \geq \alpha$. Using the fact that Ω is contained in the half-space $\{z \in \mathbb{R}^n : (z - x, \nu(\tilde x)) \leq |\tilde x - x|\}$ (due to the convexity of Ω), we obtain

$$(y(\eta) - x, \eta) \leq (y(\eta) - x, \nu(\tilde x)) + |y(\eta) - x||\eta - \nu(\tilde x)| \leq \varepsilon + \operatorname{diam}(\Omega)\sqrt{2(1-\alpha)} \leq \lambda_0/2,$$

provided α is close to 1 and ε is small enough, say $1 - \alpha + \varepsilon < \varepsilon_0 = \varepsilon_0(\Omega, \lambda_0)$. This along with (13.6) implies

$$\{x - \lambda\eta \in \mathbb{R}^n : \lambda \in [0, \lambda_0]\} \subset \Sigma(y(\eta), \lambda_0).$$

It then follows from (13.4) that $[0,\varepsilon] \ni \lambda \mapsto u(x-\lambda\eta)$ is nondecreasing for any $\eta \in S^{n-1}$ satisfying $(\eta, \nu(\tilde{x})) \geq \alpha$. This property guarantees the existence of $\gamma = \gamma(\Omega, \lambda_0) > 0$ such that

$$\left.\begin{array}{l}\text{for all } x \in \Omega_\varepsilon \text{ there exists a measurable set } I_x \subset \Omega \setminus \Omega_\varepsilon \\ \text{satisfying } \operatorname{meas} I_x \geq \gamma \ \text{ and } \ u(\xi) \geq u(x) \text{ for all } \xi \in I_x.\end{array}\right\} \tag{13.7}$$

Indeed (decreasing the value of ε if necessary), it is sufficient to take a conical piece

$$I_x = \Omega_\varepsilon^c \cap \{x - \lambda\eta : \eta \in S^{n-1},\ (\eta, \nu(\tilde{x})) \geq \alpha,\ \lambda \in [0, \lambda_0]\}.$$

Since $\varphi_1 \geq C_\varepsilon$ on $\Omega \setminus \Omega_\varepsilon$ for some $C_\varepsilon > 0$, we deduce from (13.7) that

$$C_\varepsilon \gamma u(x) \leq C_\varepsilon \int_{I_x} u(\xi)\, d\xi \leq \int_{I_x} u(\xi)\varphi_1(\xi)\, d\xi \leq \int_\Omega u(\xi)\varphi_1(\xi)\, d\xi$$

and the lemma is proved. □

Proof of Theorem 13.1. First assume (i). The proof will consist of the following four steps:

1. $\int_\Omega u\delta\, dx \leq C$, $\int_\Omega |f(u)|\delta\, dx \leq C$, where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$,
2. $u + |\nabla u| \leq C$ in a neighborhood of $\partial\Omega$,
3. $\|\nabla u\|_2 \leq C$,
4. $\|u\|_\infty \leq C$.

Step 1. This step is almost the same as Step 1 in the proof of Theorem 10.1 and we leave the detailed proof to the reader.

Step 2. Since Ω is convex and smooth, we can find $\lambda_0, c_0 > 0$ such that

$$\Sigma'(y,\lambda) \subset \Omega, \quad \lambda \leq \lambda_0 \quad \text{and} \quad (\nu(x), \nu(y)) > c_0, \quad x \in \partial\Sigma(y, \lambda_0) \cap \partial\Omega.$$

We shall now apply the moving planes method (cf. [239], [183]) to show that

$$u(R(y,\lambda)x) \geq u(x), \qquad y \in \partial\Omega,\ x \in \Sigma(y,\lambda),\ \lambda \leq \lambda_0. \tag{13.8}$$

Without loss of generality, we may assume that $y = 0$ and that $\nu(0) = -e_1$ (in particular, Ω lies entirely in the upper half-space $\{x_1 > 0\}$). For each $x = (x_1, x')$, we denote $x^\lambda := R(0,\lambda)x = (2\lambda - x_1, x')$, $\Sigma_\lambda := \Sigma(0,\lambda) = \Omega \cap \{x_1 < \lambda\}$, and $\Sigma'_\lambda := \Sigma'(0,\lambda)$. Define

$$w^\lambda(x) = u(x^\lambda) - u(x), \qquad \text{for } x \in \Sigma_\lambda,\ 0 < \lambda \leq \lambda_0,$$

and set

$$E := \bigl\{\mu \in (0, \lambda_0] : w^\lambda(x) \geq 0 \text{ for all } x \in \Sigma_\lambda \text{ and } \lambda \in (0, \mu)\bigr\}.$$

Since $\frac{\partial u}{\partial x_1}(0) > 0$ by Hopf's lemma, we have $\lambda \in E$ for $\lambda > 0$ small. Assume for contradiction that $\bar\lambda := \sup E < \lambda_0$. We have

$$w^\lambda \geq 0, \qquad \text{for all } x \in \Sigma_\lambda \text{ and } \lambda \in (0, \bar\lambda], \tag{13.9}$$

and there exists a sequence $\lambda_i \to \bar\lambda$, with $\bar\lambda < \lambda_i < \lambda_0$, such that $\min_{\overline{\Sigma_{\lambda_i}}} w^{\lambda_i} < 0$. Since $w^\lambda = 0$ on $\{x_1 = \lambda\} \cap \overline{\Omega}$ and

$$w^\lambda > 0 \ \text{ on } \{x_1 < \lambda\} \cap \partial\Omega, \quad \text{for all } \lambda < \lambda_0, \tag{13.10}$$

it follows that this minimum is attained at a point $q_i \in \Sigma_{\lambda_i}$. Therefore $\nabla w^{\lambda_i}(q_i) = 0$. On the other hand, since $\frac{\partial u}{\partial x_1} = (e_1 \cdot \nu)\frac{\partial u}{\partial \nu} \geq c > 0$ on $\{x_1 \leq \lambda_0\} \cap \partial\Omega$ and

$$w^\lambda(x) = u(2\lambda - x_1, x') - u(x_1, x') = 2(\lambda - x_1)\frac{\partial u}{\partial x_1}(\xi(x)),$$

with $|\xi(x) - x| \leq 2(\lambda - x_1)$, we see that $w^\lambda(x) \geq 0$ for x in an ε-neighborhood of $\{x_1 = \lambda\} \cap \partial\Omega$, with $\varepsilon > 0$ independent of $\lambda \in (0, \lambda_0]$. Therefore, we may assume that $q_i \to \bar q \in \overline{\Sigma_{\bar\lambda}}$, $\bar q \notin \{x_1 = \bar\lambda\} \cap \partial\Omega$, and by continuity we get

$$w^{\bar\lambda}(\bar q) = 0 \quad \text{and} \quad \nabla w^{\bar\lambda}(\bar q) = 0. \tag{13.11}$$

But (13.9) implies

$$-\Delta w^{\bar\lambda}(x) = f\bigl(u(x^{\bar\lambda})\bigr) - f(u(x)) \geq -cw^{\bar\lambda}(x) \quad \text{and} \quad w^{\bar\lambda}(x) \geq 0, \qquad x \in \Sigma_{\bar\lambda},$$

for some constant $c > 0$ (depending on u). By Hopf's lemma (cf. Proposition 52.1 and Remark 52.2), this along with (13.11) implies $w^{\bar\lambda} = 0$ in $\Sigma_{\bar\lambda}$, contradicting (13.10). Consequently, $\bar\lambda = \lambda_0$, which proves (13.8). This guarantees that u satisfies (13.4). By Lemma 13.2 and Step 1, we deduce that $u \leq C$ on Ω_ε for some $\varepsilon, C > 0$ depending only on Ω. Now the bound for ∇u in $\Omega_{\varepsilon/2}$ follows from interior-boundary elliptic L^p-estimates (see Appendix A) and the embedding $W^{2,p} \hookrightarrow C^1$ for $p > n$. In particular, we have shown that

$$\left|\frac{\partial u}{\partial \nu}\right| \leq C, \quad x \in \partial\Omega. \tag{13.12}$$

Step 3. Notice that Steps 1 and 2 imply

$$\|f(u)\|_1 \leq C. \tag{13.13}$$

First consider the case $n \geq 3$. The Hölder and Sobolev inequalities and (13.13) guarantee

$$\int_\Omega u^2|f(u)|^{2/n}\, dx \leq \|u\|_{2^*}^2 \|f(u)\|_1^{2/n} \leq C\|\nabla u\|_2^2.$$

Pohozaev's identity (5.1) and (13.12) yield

$$\Big|\int_\Omega |\nabla u|^2\,dx - 2^*\int_\Omega F(u)\,dx\Big| \le C.$$

Since $\int_\Omega |\nabla u|^2\,dx = \int_\Omega uf(u)\,dx$, the last two estimates and (13.2) imply

$$\begin{aligned}2^*\int_\Omega F(u)\,dx &\le \int_\Omega uf(u)\,dx + C \le \theta\int_\Omega F(u)\,dx + \varepsilon\int_\Omega u^2|f(u)|^{2/n}\,dx + C_\varepsilon \\ &\le (\theta+\varepsilon C)\int_\Omega F(u)\,dx + \tilde C_\varepsilon.\end{aligned}$$

Choosing $\varepsilon < (2^*-\theta)/C$ we obtain $\int_\Omega F(u)\,dx \le C$, hence $\|\nabla u\|_2 \le C$.

Next let $n=2$. Set $\gamma := 1-1/\sigma$. Given $\varepsilon>0$, the assumption $\lim_{u\to\infty} f(u)/u^\sigma = 0$ guarantees the existence of $C_\varepsilon>0$ such that

$$uf(u) \le \varepsilon u^2 f(u)^\gamma + C_\varepsilon.$$

Similarly as above we obtain

$$\begin{aligned}\|\nabla u\|_2^2 &= \int_\Omega uf(u)\,dx \le \varepsilon\int_\Omega u^2|f(u)|^\gamma\,dx + C_\varepsilon \\ &\le \varepsilon\|u\|^2_{2/(1-\gamma)}\|f(u)\|_1^\gamma \le \varepsilon C\|\nabla u\|_2^2 + C_\varepsilon,\end{aligned}$$

which proves the assertion.

Step 4. If

$$f(u) \le C(1+u^p) \quad \text{for some } p<p_S \tag{13.14}$$

(which is always true if $n=2$), then one can use standard bootstrap estimates based on L^q-estimates (see Appendix A) to show that the $W^{1,2}$-bound from Step 3 guarantees an L^∞-bound. If $n\ge 3$ and (13.14) is not true, then we use the following estimates (see [96] and cf. the proof of Proposition 3.3).

Let $p>1$, $a_p := (p+1)^2/4$ and $q := (p+1)n/(n-2)$. Then

$$\begin{aligned}\Big(\int_\Omega u^q\,dx\Big)^{(n-2)/n} &= \|u^{(p+1)/2}\|^2_{2^*} \le C\int_\Omega |\nabla u^{(p+1)/2}|^2\,dx = Ca_p\int_\Omega |\nabla u|^2 u^{p-1}\,dx \\ &= C\frac{a_p}{p}\int_\Omega f(u)u^p\,dx \le \varepsilon\int_\Omega u^{p+\sigma}\,dx + C_\varepsilon,\end{aligned}$$

where $\sigma = (n+2)/(n-2)$. Next Hölder's inequality and Step 3 yield

$$\begin{aligned}\int_\Omega u^{p+\sigma}\,dx &= \int_\Omega u^{q(n-2)/n+4/(n-2)}\,dx \le \Big(\int_\Omega u^q\,dx\Big)^{(n-2)/n}\Big(\int_\Omega u^{2^*}\,dx\Big)^{2/n} \\ &\le C\Big(\int_\Omega u^q\,dx\Big)^{(n-2)/n}.\end{aligned}$$

These estimates imply $\|u\|_q \le C$, hence $\|f(u)\|_{q/\sigma} \le C$. Since q can be made arbitrarily large, the L^p-estimates (see Appendix A) conclude the proof in case (i).

Next consider assumption (ii). Instead of Ω being convex we now assume (13.3). Since the convexity assumption was used only in the proof of Step 2, it is sufficient to modify the proof of this step. Choose $x_0\in\partial\Omega$. Then there exists a ball $B_r \subset \mathbb{R}^n\setminus\Omega$ of radius r such that $x_0\in\partial B_r$. The radius r can be chosen independent of x_0 and, without loss of generality, we may assume $r=1$. Choose a coordinate system such that B_r is centered at the origin and $x_0=(1,0,\dots,0)$. Set $y=J(x) := x/|x|^2$ and $w(y) = |x|^{n-2}u(x)$. Then

$$-\Delta w(y) = g(y,w) \quad \text{in } \mathcal{O} := J(\Omega),$$

where $g(y,w) := f(|y|^{n-2}w)/|y|^{n+2}$ is nonincreasing in y due to (13.3). Since $\mathcal{O}\subset B_r$ is smooth and $x_0\in\partial\mathcal{O}\cap\partial B_r$ we can use the moving planes method in order to get the existence of $\varepsilon_{x_0},\gamma_{x_0}>0$ with the following property: for any $y\in\mathcal{O}$, $|y-x_0|<\varepsilon_{x_0}$, there exists a set $K_y \subset \{z\in\mathcal{O} : \operatorname{dist}(z,\partial\mathcal{O}) > \varepsilon_{x_0}\}$ satisfying meas $K_y \ge \gamma_{x_0}$ and $w(\xi)\ge w(y)$ for all $\xi\in K_y$. Going back to the original variables and using the compactness of $\partial\Omega$ we get the existence of $\varepsilon,\gamma,c>0$ such that (13.7) is true, with $u(\xi)\ge u(x)$ replaced by $u(\xi)\ge cu(x)$. The rest of the proof of Step 2 is the same as in case (i).

Finally consider case (iii). Then Steps 1 and 4 can be proved in the same way as in case (i). Repeating the arguments in the proof of Step 2 of case (i) we obtain a uniform bound for u and $|\nabla u|$ in a neighborhood of Γ_1. Without loss of generality we may assume $x_0=0$, hence $x\cdot\nu(x)\le 0$ for all $x\in\Gamma_2$. These facts and Pohozaev's identity (5.1) imply

$$2^*\int_\Omega F(u)\,dx - \int_\Omega uf(u)\,dx \le C. \tag{13.15}$$

Next using Lemma 50.4 with $\tau := 1/(n+1)$ and $q := 2(n+1)/(n-1)$, Step 1 and Hölder's inequality, we obtain

$$\begin{aligned}\int_\Omega uf(u)\,dx &= \|\nabla u\|_2^2 \ge c_1\Big\|\frac{u}{\delta^\tau}\Big\|_q^2 \ge c_2\Big\|\frac{u}{\delta^\tau}\Big\|_q^2\|f(u)\delta\|_1^{1-2/q} \\ &\ge c_2\int_\Omega \frac{u^2}{\delta^{2\tau}}\big(|f(u)|\delta\big)^{1-2/q}\,dx = c_2\int_\Omega u^2|f(u)|^{2/(n+1)}\,dx.\end{aligned}$$

Now (13.2) with $\kappa = 2/(n+1)$, (13.15) and the last estimate imply

$$\begin{aligned}\int_\Omega uf(u)\,dx &\le \theta\int_\Omega F(u)\,dx + \varepsilon\int_\Omega u^2|f(u)|^{2/(n+1)}\,dx + C_\varepsilon \\ &\le (\theta/2^*+\varepsilon C)\int_\Omega uf(u)\,dx + C_\varepsilon\end{aligned}$$

and the choice of ε small enough concludes the proof. $\square$

The following corollary can be proved in the same way as Corollary 10.3.

Corollary 13.3. *Let* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy the assumptions in Theorem* 13.1 *and* $\limsup_{u\to 0+} f(u)/u < \lambda_1$. *Then problem* (2.1) *possesses at least one positive classical solution.*

Remark 13.4. If one is interested only in the existence of positive solutions of (2.1) without knowing their a priori bounds, then the technical assumption (13.2) can be omitted, see [183]. The proof is based on an approximation of the function f, on the mountain pass theorem (including uniform bounds for the energy of approximating solutions) and Pohozaev's identity. □

Chapter II

Model Parabolic Problems

14. Introduction

In Chapter II, we mainly consider semilinear parabolic problems of the form

$$\left.\begin{aligned} u_t - \Delta u &= f(u), && x \in \Omega,\ t > 0, \\ u &= 0, && x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned}\right\} \tag{14.1}$$

where f is a C^1-function with a superlinear growth. For simplicity, we formulate most of our assertions for the model case $f(u) = |u|^{p-1}u$ with $p > 1$, but the methods of our proofs can be applied to more general parabolic problems (not necessarily of the form (14.1)). Some of possible generalizations and modifications will be mentioned as remarks, other can be found in the subsequent chapters.

15. Well-posedness in Lebesgue spaces

Definition 15.1. Given a Banach space X of functions defined in Ω, $u_0 \in X$ and $T \in (0,\infty]$, we say that the function $u \in C([0,T),X)$ is a **solution** (more precisely, a **classical** X**-solution**) of (14.1) in $[0,T)$ if $u \in C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times (0,T))$, $u(0) = u_0$ and u is a classical solution of (14.1) for $t \in (0,T)$. If Ω is unbounded, then we also require $u \in L^\infty_{loc}((0,T), L^\infty(\Omega))$.

If $X = L^\infty(\Omega)$, then, instead of the condition $u \in C([0,T),X)$, we require $u \in C((0,T),X)$ and $\|u(t) - e^{-tA}u_0\|_\infty \to 0$ as $t \to 0$, where e^{-tA} is the Dirichlet heat semigroup in Ω (cf. Appendix B).

We say that (14.1) is **well-posed** in X if, given $u_0 \in X$, there exist $T > 0$ and a unique classical X-solution of (14.1) in $[0,T]$. □

It is well known that (14.1) is well-posed in $X = W_0^{1,q}(\Omega)$ for any $q > n$ if Ω is bounded, or in $X = L^\infty(\Omega)$ for any Ω (see Example 51.9 and Remark 51.11). In this section we study the well-posedness of the model problem

$$\left.\begin{aligned} u_t - \Delta u &= |u|^{p-1}u, && x \in \Omega,\ t > 0, \\ u &= 0, && x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned}\right\} \tag{15.1}$$