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Everywhere regularity of functionals with φ-growth

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Abstract. We prove $C^{1,\alpha}$-regularity for local minimizers of functionals with φ-growth, giving also the decay estimate. In particular, we present a unified approach in the case of power-type functions.

1. Introduction

Let φ be a convex, C^1-function and consider the functional:

$$\mathcal{F}(\mathbf{u}) = \int_\Omega \varphi(|\nabla \mathbf{u}|)\, dx \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and $\mathbf{u} : \Omega \to \mathbb{R}^N$.

The standard examples for convex functions φ are

$$\varphi(t) = \int_0^t (\kappa + s^2)^{\frac{p-2}{2}}\, s\, ds \qquad \text{and} \qquad \varphi(t) = \int_0^t (\kappa + s)^{p-2}\, s\, ds,$$

where $\kappa \geq 0$.

We say that $\mathbf{u}$ is a local minimizer for F if

$$\mathcal{F}(\mathbf{u}, \operatorname{spt} \mathbf{v}) \leq \mathcal{F}(\mathbf{u} + \mathbf{v}, \operatorname{spt} \mathbf{v}) \quad \forall \mathbf{v} \in C_0^1(\Omega).$$

The associated Euler Lagrange system is

$$-\operatorname{div}\left(\varphi'(|\nabla \mathbf{u}|)\frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|}\right) = 0 \tag{1.2}$$

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In a fundamental paper Uhlenbeck [26] proved everywhere $C^{1,\alpha}$-regularity for local minimizers of the p-growth functional with $p \geq 2$. Later on a large number of generalizations have been made. The case $1 < p < 2$ was considered by Acerbi and Fusco [1] where also the dependence of the functional from x and $\mathbf{u}$ was investigated, (see [23] for a complete list).

Lieberman [18] generalized the regularity theory of Ladyzhenskaya and Uraltseva for equations with φ-growth. Lipschitz regularity results for systems or functionals with nonstandard growth conditions have been considered by Marcellini [19–21] and Esposito et al. [10, 11]. We refer to a recent book of Bildhauer [3] for a general treatment.

In a recent paper [22] Marcellini and Papi proved Lipschitz regularity for local minimizers of functionals with growth conditions general enough to embrace linear and exponential ones. A general approach in order to get $C^{1,\alpha}$-regularity for systems is to prove first Lipschitz continuity and then, using the C^1-property of the operator, conclude with the help of classical results. Another approach is contained in a paper of Esposito and Mingione [12] in which they raised the question of proving $C^{1,\alpha}$-regularity of φ-growth by comparison with powers.

Unfortunately, this is not enough to get an excess decay estimate out of the power case.

For this reason, our goal is to prove the $C^{1,\alpha}$-regularity for functionals with φ-growth giving the decay estimate of the excess functional:

$$\Phi(\mathbf{u}, B) = \fint_B |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u}) \rangle_B|^2 \, dx \tag{1.3}$$

where $\mathbf{V}(\mathbf{Q}) = \sqrt{\varphi'(|\mathbf{Q}|)/|\mathbf{Q}|}\,\mathbf{Q}$ and $B \subset \Omega$ is a ball. To this aim, we make suitable assumptions on the function φ in order to ensure the continuity of the second derivatives of φ. In particular, the case of slow growth [13] and fast growth are ruled out.

Our main theorem is the following:

Theorem 1.1. *Let* $\mathbf{u} \in W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ *be a local minimizer of* (1.1)*, where* φ *satisfies Assumption* 2.2*. Then* $\mathbf{V}(\nabla \mathbf{u})$ *and* $\nabla \mathbf{u}$ *are locally* α*-Hölder continuous for some* $\alpha > 0$.

We present a unified approach to the superquadratic and subquadratic p-growth, also considering more general functions than the powers.

The results presented here rely on some technical lemmas that have been proved in a paper of Diening and Ettwein [6], where they get fractional estimates for non-differentiable elliptic systems with φ-growth.

2. Notation and preliminaries

To simplify the notation, the letter c will denote any positive constant, which may vary throughout the paper. For $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a ball $B \subset \mathbb{R}^n$ we define

$$\langle w \rangle_B := \fint_B w(x)\, dx := \frac{1}{|B|} \int_B w(x)\, dx, \tag{2.1}$$

where $|B|$ is the n-dimensional Lebesgue measure of B. For $\lambda > 0$ we denote by λB the ball with the center as B but λ-times the radius. By $e_1, \dots, e_n$ we denote the unit vectors of $\mathbb{R}^n$. For $U, \Omega \subset \mathbb{R}^n$ we write $U \Subset \Omega$ if the closure of U is a compact subset of Ω. We define $\delta_{i,j} := 0$ for $i \neq j$ and $\delta_{i,i} = 1$.

The following definitions and results are standard in the context of N-functions. A real function $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is said to be an N-function if it satisfies the following conditions: $\varphi(0) = 0$ and there exists the derivative φ' of φ. This derivative is right continuous, non-decreasing and satisfies $\varphi'(0) = 0$, $\varphi'(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} \varphi'(t) = \infty$. Especially, φ is convex.

We say that φ satisfies the Δ_2-condition, if there exists $c_1 > 0$ such that for all $t \geq 0$ holds $\varphi(2t) \leq c_1\, \varphi(t)$. By $\Delta_2(\varphi)$ we denote the smallest constant c_1. Since $\varphi(t) \leq \varphi(2t)$ the Δ_2 condition is equivalent to $\varphi(2t) \sim \varphi(t)$. For a family $\{\varphi_\lambda\}_\lambda$ of N-functions we define $\Delta_2(\{\varphi_\lambda\}_\lambda) := \sup_\lambda \Delta_2(\varphi_\lambda)$.

By L^φ and $W^{1,\varphi}$ we denote the classical Orlicz and Sobolev–Orlicz spaces, i.e. $f \in L^\varphi$ iff $\int \varphi(|f|)\, dx < \infty$ and $f \in W^{1,\varphi}$ iff $f, \nabla f \in L^\varphi$. By $W_0^{1,\varphi}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,\varphi}(\Omega)$.

By $(\varphi')^{-1} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ we denote the function

$$(\varphi')^{-1}(t) := \sup\,\{s \in \mathbb{R}^{\geq 0} : \varphi'(s) \leq t\}.$$

If φ' is strictly increasing then $(\varphi')^{-1}$ is the inverse function of φ'. Then $\varphi^* : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)\, ds$$

is again an N-function and $(\varphi^*)'(t) = (\varphi')^{-1}(t)$ for $t > 0$. It is the complementary function of φ. Note that $\varphi^*(t) = \sup_{s\geq 0}(st - \varphi(s))$ and $(\varphi^*)^* = \varphi$. For all $\delta > 0$ there exists c_δ (only depending on $\Delta_2(\{\varphi, \varphi^*\})$ such that for all $t, s \geq 0$ holds

$$ts \leq \delta\varphi(t) + c_\delta\varphi^*(s). \tag{2.2}$$

For $\delta = 1$ we have $c_\delta = 1$. This inequality is called Young 's inequality. For all $t \geq 0$

$$\begin{aligned} &\frac{t}{2}\varphi'\Big(\frac{t}{2}\Big) \leq \varphi(t) \leq t\,\varphi'(t),\\ &\varphi\bigg(\frac{\varphi^*(t)}{t}\bigg) \leq \varphi^*(t) \leq \varphi\bigg(\frac{2\,\varphi^*(t)}{t}\bigg). \end{aligned} \tag{2.3}$$

Therefore, uniformly in $t \geq 0$

$$\varphi(t) \sim \varphi'(t)t, \quad \varphi^*\big(\varphi'(t)\big) \sim \varphi(t), \tag{2.4}$$

where the constants only depend on $\Delta_2(\{\varphi, \varphi^*\})$. If $\rho(t) = a\varphi(b\,t)$ for some $a, b > 0$ and all $t \geq 0$, then

$$\rho^*(t) = a\,\varphi^*\Big(\tfrac{t}{a\,b}\Big). \tag{2.5}$$

If φ and ρ are N-functions with $\varphi(t) \leq \rho(t)$ for all $t \geq 0$, then

$$\rho^*(t) \leq \varphi^*(t) \tag{2.6}$$

for all $t \geq 0$.

Throughout the paper we will assume φ satisfies the following assumption.

Assumption 2.1. Let φ be an N-function such that φ is C^1 on $[0, \infty)$ and C^2 on $(0, \infty)$. Further assume that

$$\varphi'(t) \sim t\varphi''(t) \tag{2.7}$$

uniformly in $t > 0$.

We remark that under these assumptions $\Delta_2(\{\varphi, \varphi^*\}) < \infty$ will be automatically satisfied, where $\Delta_2(\{\varphi, \varphi^*\})$ depends only on the constant in (2.7). In the proof of the regularity Theorem we will additionally require that φ'' is Hölder continuous away from zero.

Assumption 2.2. Let φ be as in Assumption 2.1 such that there exists $\beta \in (0, 1]$ and $c > 0$ such that

$$\left|\varphi''(s+t) - \varphi''(t)\right| \leq c\,\varphi''(t)\left(\tfrac{|s|}{t}\right)^{\beta} \tag{2.8}$$

for all $t > 0$ and $s \in \mathbb{R}$ with $|s| < \frac{1}{2}t$.

Remark 2.3. Let φ satisfy Assumption 2.1. Further, let $t > 0$ and $s \in \mathbb{R}$ with $|s| < \frac{1}{2}t$. Then by Taylor's formula, $|s+t| \sim t$ and $\Delta_2(\varphi) < \infty$ get

$$\left|\varphi'(s+t) - \varphi'(t)\right| \leq c\,\varphi''(s+t)\,|s| \leq c\,\tfrac{\varphi'(s+t)}{s+t}\,|s| \leq c\,\varphi'(t)\tfrac{|s|}{t}.$$

So φ' is Lipschitz continuous away from zero. Compare this with (2.8).

We notice that assumption (2.8) is satisfied for example in all of the following three cases.

$$\begin{aligned} \varphi(t) &= t^p, \\ \varphi(t) &= t^p \log^{\beta}(e+t), \qquad \beta > 0, \\ \varphi(t) &= t^p \log\log(e+t), \end{aligned}$$

with $1 < p < \infty$.

For given φ we define the associated N-function ψ by

$$\psi'(t) := \sqrt{\varphi'(t)t}. \tag{2.9}$$

Note that

$$\psi''(t) = \frac{1}{2}\left(\frac{\varphi''(t)}{\varphi'(t)}t + 1\right)\sqrt{\frac{\varphi'(t)}{t}} = \frac{1}{2}\left(\frac{\varphi''(t)}{\varphi'(t)}t + 1\right)\frac{\psi'(t)}{t}. \tag{2.10}$$

It is shown in [6, Lemma 25] that if φ satisfies Assumption 2.1, then also φ^*, ψ, and ψ^* satisfy this assumption and $\psi''(t) \sim \sqrt{\varphi''(t)}$.

Define $\mathbf{A}, \mathbf{V} : \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ in the following way:

$$\mathbf{A}(\mathbf{Q}) = \varphi'(|\mathbf{Q}|)\frac{\mathbf{Q}}{|\mathbf{Q}|}, \tag{2.11a}$$

$$\mathbf{V}(\mathbf{Q}) = \psi'(|\mathbf{Q}|)\frac{\mathbf{Q}}{|\mathbf{Q}|}. \tag{2.11b}$$

For $\lambda \geq 0$ we define the *shifted N-function* φ_λ by $\varphi_\lambda(t) = \int_0^t \varphi_\lambda'(s)\,ds$ with

$$\varphi_\lambda'(t) := \frac{\varphi'(\lambda + t)}{\lambda + t}\,t \tag{2.12}$$

for $t > 0$. The shifted N-functions have been introduced in [6]. In [8] and [7] they have been used in the a priori and a posteriori analysis of finite element approximations of (1.1). See [24] for a detailed study of the shifted N-functions.

The connection between $\mathbf{A}$, $\mathbf{V}$, and $\{\varphi_\lambda\}_{\lambda\geq 0}$ is best reflected in the following lemma.

Lemma 2.4. *Let φ satisfy Assumption 2.1 and let $\mathbf{A}$ and $\mathbf{V}$ be defined by (2.11). Then*

$$\big(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})\big)\cdot\big(\mathbf{P} - \mathbf{Q}\big) \sim \big|\mathbf{V}(\mathbf{P}) - \mathbf{V}(\mathbf{Q})\big|^2 \tag{2.13a}$$

$$\sim \varphi_{|\mathbf{P}|}(|\mathbf{P} - \mathbf{Q}|), \tag{2.13b}$$

$$\sim |\mathbf{P} - \mathbf{Q}|^2\,\varphi''\big(|\mathbf{P}| + |\mathbf{Q}|\big), \tag{2.13c}$$

and

$$\big|\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})\big| \sim \varphi_{|\mathbf{P}|}'\big(|\mathbf{P} - \mathbf{Q}|\big) \tag{2.13d}$$

$$\sim \varphi''\big(|\mathbf{P}| + |\mathbf{Q}|\big)\,|\mathbf{P} - \mathbf{Q}| \tag{2.13e}$$

uniformly in $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{N\times n}$. Moreover,

$$\mathbf{A}(\mathbf{Q})\cdot\mathbf{Q} \sim |\mathbf{V}(\mathbf{Q})|^2 \sim \varphi(|\mathbf{Q}|), \tag{2.13f}$$

$$|\mathbf{A}(\mathbf{Q})| \sim \varphi'(|\mathbf{Q}|) \tag{2.13g}$$

uniformly in $\mathbf{Q} \in \mathbb{R}^{N\times n}$.

Note that if $\varphi''(0)$ does not exists, the expression in (2.13c) and (2.13e) are continuously extended by zero for $|\mathbf{P}| = |\mathbf{Q}| = 0$.

Proof. The lemma is a direct consequence of Lemma 3 and Lemma 21 of [6]. □

Remark 2.5. By definition of φ_λ if follows directly, that $(\varphi_\lambda)_{\lambda_2} = \varphi_{\lambda+\lambda_2}$ for $\lambda, \lambda_2 \geq 0$.

It has been proved in [6] that the functions φ_λ with $\lambda \geq 0$ share the same properties of φ. In particular, we have the following result.

Lemma 2.6. *Let φ satisfy Assumption 2.1. Then for all $\lambda \geq 0$ the function φ_λ satisfies Assumption 2.1 and $\Delta_2(\{\varphi_\lambda\}_{\lambda\geq 0}, \{(\varphi_\lambda)^*\}_{\lambda\geq 0}) < \infty$. Moreover,*

$$\varphi_\lambda''(t) \sim \varphi''(\lambda + t) \sim \frac{\varphi'(\lambda + t)}{\lambda + t} = \frac{\varphi_\lambda'(t)}{t} \tag{2.14}$$

uniformly in $\lambda, t \geq 0$ with $\lambda + t > 0$. In particular, φ_λ satisfies Assumption 2.1 with constants independent of $\lambda \geq 0$. Moreover, for $t > 0$ holds

$$\varphi_\lambda''(t) = \frac{\varphi''(\lambda + t)\, t}{\lambda + t} + \frac{\varphi'(\lambda + t)\, \lambda}{(\lambda + t)^2}. \tag{2.15}$$

If $\lambda > 0$, then φ_λ is C^2 on $[0, \infty)$.

If φ satisfies Assumption 2.2, then φ_λ satisfies Assumption 2.2 with $\beta > 0$ and the constant in (2.8) does not depend on λ.

Proof. Let φ satisfy Assumption 2.1 The formula for φ_λ'' follows directly from the definition of φ_λ. This implies the C^2-property on $[0, \infty)$ for $\lambda > 0$. The other claims have been proved in [6, Lemma 24 and 27].

Assume now that φ satisfies Assumption 2.2 with $\beta > 0$. Then it follows after a short computation from (2.15), (2.8), and Remark 2.3 that (2.8) holds for all φ_λ with the same β such that the constant does not depend on $\lambda \geq 0$. ◻

We state a generalization of Lemma 2.1 of [1] to the context of convex functions φ.

Lemma 2.7. [6, Lemma 20] *Let φ be an N-function with $\Delta_2(\{\varphi, \varphi^*\}) < \infty$. Then uniformly for all $\mathbf{P}_0, \mathbf{P}_1 \in \mathbb{R}^{N\times n}$ with $|\mathbf{P}_0| + |\mathbf{P}_1| > 0$ holds*

$$\int_0^1 \frac{\varphi'(|\mathbf{P}_\theta|)}{|\mathbf{P}_\theta|}\, d\theta \sim \frac{\varphi'(|\mathbf{P}_0| + |\mathbf{P}_1|)}{|\mathbf{P}_0| + |\mathbf{P}_1|}, \tag{2.16}$$

where $\mathbf{P}_\theta := (1-\theta)\mathbf{P}_0 + \theta\mathbf{P}_1$. The constants only depend on $\Delta_2(\{\varphi, \varphi^*\})$.

Remark 2.8. Some of the results in this paper are stated under the condition that $\Delta_2(\{\varphi, \varphi^*\}) < \infty$ or that φ satisfies Assumption 2.1. Due to Lemma 2.7 all of these results remain valid for φ_λ with $\lambda \geq 0$ as well.

The following result is contained in [6, Lemma 31].

Lemma 2.9. *Let φ be an N-function with $\Delta_2(\{\varphi, \varphi^*\}) < \infty$. Then there exist $q_2 > 1$, $c > 0$ which only depend on $\Delta_2(\{\varphi, \varphi^*\})$ such that*

$$\varphi_\lambda(\theta t) \leq c\theta^{q_2}\varphi_\lambda(t) \tag{2.17}$$

for all $t, \lambda \geq 0$ and all $\theta \in [0, 1]$.

Since $(\varphi^*)'$ is the inverse of φ', it follows that

$$\mathbf{A}^{-1}(\mathbf{Q}) = (\varphi^*)'(|\mathbf{Q}|)\tfrac{\mathbf{Q}}{|\mathbf{Q}|} \tag{2.18}$$

for all $\mathbf{Q} \in \mathbb{R}^{N\times n}$. In particular, $\mathbf{A}$ is invertible. The same holds for $\mathbf{A}_\lambda$ and $\mathbf{V}_\lambda$.

The following result generalizes Lemma 3 of [9] to the context of convex functions.

Lemma 2.10. *Let φ satisfy Assumption 2.1. Then there exists $\beta > 0$, which only depends on the constant in (2.7), such that φ', $(\varphi^*)'$, ψ', and $(\psi^*)'$ are β-Hölder continuous on* $[0,\infty)$ *and* $\mathbf{A}$, $\mathbf{A}^{-1}$, $\mathbf{V}$, *and* $\mathbf{V}^{-1}$ *are β-Hölder continuous on* $\mathbb{R}^{N\times n}$.

Proof. Let q_2 be as in Lemma 2.9, then q_2 only depends on $\Delta_2(\{\varphi,\varphi^*\})$. We will show that φ' is β-Hölder continuous, with $\beta := q_2 - 1$. Let $a, b \in [0,\infty)$ with $|a-b| \le 1$. Then by (2.13d) applied to the case $n = 1$ we get

$$|\varphi'(a) - \varphi'(b)| \le c\,\varphi_b'(|a-b|).$$

Now, with $t\varphi'(t) \sim \varphi(t)$ and (2.17) follows

$$|\varphi'(a) - \varphi'(b)| \le c\,\frac{\varphi_b(|a-b|)}{|a-b|} \le c\,|a-b|^{q_2-1}\varphi_b(1).$$

This proves that φ' is β-Hölder continous, where $\beta > 0$ only depends on $\Delta_2(\{\varphi,\varphi^*\})$.

Due to [6, Lemma 25] also φ^* and ψ satisfy Assumption 2.1. Thus, we see that also $(\varphi^*)'$, ψ', and $(\psi^*)'$ are α-Hölder continuous, where $\alpha > 0$ only depends on the constant in (2.7). Now, the definition of $\mathbf{A}$ and $\mathbf{V}$ and (2.18) imply that $\mathbf{A}$, $\mathbf{A}^{-1}$, $\mathbf{V}$, and $\mathbf{V}^{-1}$ are α-Hölder continuous. □

Remark 2.11. Due to Lemma 2.6 it is possible to apply Lemma 2.10 also to the shifted-versions uniformly in $\lambda \ge 0$. For example there exists $\beta > 0$ such that $\mathbf{V}_\lambda^{-1}$ is β-Hölder continuous for all $\lambda \ge 0$.

Remark 2.12. Let φ satisfy Assumption 2.1. Then, for all $\mathbf{Q}, \mathbf{H} \in \mathbb{R}^{N\times n}$ with $\mathbf{Q} \ne 0$ holds

$$\sum_{ij\alpha\gamma} \partial_{\alpha\gamma} A_{ij}(\mathbf{Q}) H_{ij} H_{\alpha\gamma} \sim \varphi''(|\mathbf{Q}|)\,|\mathbf{H}|^2.$$

This follows easily from

$$\partial_{\alpha\gamma} A_{ij}(\mathbf{Q}) = \tfrac{\varphi'(|\mathbf{Q}|)}{|\mathbf{Q}|}\delta_{i,j}\delta_{\alpha,\gamma} + \Big(\varphi''(|\mathbf{Q}|) - \tfrac{\varphi'(|\mathbf{Q}|)}{|\mathbf{Q}|}\Big)\frac{Q_{ij}Q_{\alpha\gamma}}{|\mathbf{Q}|^2} \tag{2.19}$$

and (2.7).

3. Caccioppoli estimates and a Gehring type result

In this paragraph we recall some estimates obtained in the paper [6] for φ-systems. Since they are local estimates, they hold true for local minimizers.

If $\mathbf{u}$ is a local minimizer of the functional 1.1, then $\mathbf{u} = (u_1, \dots, u_N)$ solves

$$-\operatorname{div}\big(\mathbf{A}(\nabla\mathbf{u})\big) = 0. \tag{3.1}$$

In other words for all test function $\boldsymbol{\xi} \in C_0^\infty(\Omega)$

$$\int \sum_{j,k} \big(A_{jk}(\nabla\mathbf{u})\big)\partial_k\xi_j\,dx = \int \sum_{j,k}\left(\varphi'(|\nabla\mathbf{u}|)\frac{\partial_k u_j}{|\nabla\mathbf{u}|}\right)\partial_k\xi_j\,dx = 0. \tag{3.2}$$

Since $\mathbf{u} \in W^{1,\varphi}_{\text{loc}}(\Omega)$, it follows that (3.2) also holds for $\boldsymbol{\xi} \in W_0^{1,\varphi}(\Omega)$.

Theorem 3.1. [6, Theorem 4 and 9][1] *Let φ satisfy Assumption 2.1 and let $\mathbf{u}$ be a local minimizer of the functional (1.1). Then there exists $K_1 > 0$ such that for all balls $B \subset \Omega$ with $2B \Subset \Omega$ holds*

$$⨍_B \varphi(|\nabla\mathbf{u}|)\,dx \le K_1 ⨍_{2B} \varphi\left(\frac{|\mathbf{u} - \langle\mathbf{u}\rangle_{2B}|}{R}\right)dx. \tag{3.3}$$

Note that K_1 only depends on the constant in (2.7).

Note that similar results as Theorem 3.1 regarding higher integrability have been proved in [5] by Cianchi and Fusco and [4] by Cianchi.

Another important tool in our proof will be the following generalization of the Poincaré inequality.

Theorem 3.2. [6, Theorem 7] *Let φ be an N-function with $\Delta_2(\{\varphi,\varphi^*\}) < \infty$. Further, let $B \subset \mathbb{R}^n$ be some ball with diameter R. Then there exist $0 < \theta < 1$ and $K > 0$, which only depend on $\Delta_2(\{\varphi,\varphi^*\})$ and R, such that for all $\mathbf{v} \in W^{1,\varphi}(B)$ holds*

$$⨍_B \varphi\left(\frac{|\mathbf{v} - \langle\mathbf{v}\rangle_B|}{R}\right)dx \le K\left(⨍_B \big(\varphi(|\nabla\mathbf{v}|)\big)^\theta\,dx\right)^{\frac{1}{\theta}}. \tag{3.4}$$

We also need the following estimate of Gehring type.

Theorem 3.3. [6, Theorem 9] *Let φ be an N-function with $\Delta_2(\{\varphi,\varphi^*\}) < \infty$ and let $\mathbf{u}$ be a local minimizer of (1.1). Then there exists $q_1 > 1$ and $c > 1$ such that for all balls B with $2B \Subset \Omega$ and all $q \in [1, q_1]$ holds*

$$\left(⨍_B |\mathbf{V}(\nabla\mathbf{u})|^{2q}\,dx\right)^{\frac{1}{q}} \le c ⨍_{2B} |\mathbf{V}(\nabla\mathbf{u})|^2\,dx \tag{3.5}$$

Especially, we have $\varphi(|\nabla\mathbf{u}|) \in L^{q_1}_{\text{loc}}(\Omega)$. The constants c and q_1 only depend on $\Delta_2(\{\varphi,\varphi^*\})$.

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[1] The original version is stated with $\langle\mathbf{u}\rangle_B$ rather than $\langle\mathbf{u}\rangle_{2B}$. The proof for $\langle\mathbf{u}\rangle_{2B}$ requires no change.

In the following we will derive an improved version of this theorem. For this we start with a reverse Hölder estimate.

Lemma 3.4. *Let φ satisfy Assumption 2.1 and let $\mathbf{u}$ be a local minimizer of the functional (1.1). Then there exist $0 < \theta < 1$ and $K_2 > 0$ such that for all balls $B \subset \Omega$ with $2B \Subset \Omega$ and all $\mathbf{Q} \in \mathbb{R}^{N\times n}$ holds*

$$\fint_B |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\,dx \leq K_2\left(\fint_{2B} |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\mathbf{Q})|^{2\theta}\,dx\right)^{\frac{1}{\theta}}. \tag{3.6}$$

Note that θ, K_2 only depend on the constant in (2.7).

Proof. Let $q_2 > 1$ be as in Lemma 2.9 and let $s := \frac{q_2}{q_2-1} > 1$. Let $\eta \in C_0^\infty(2B)$ with $\chi_B \leq \eta \leq \chi_{2B}$ and $|\nabla\eta|_\infty \leq c/R$, where R is the radius of B. Let $\boldsymbol{\xi} := \eta^s(\mathbf{u} - \mathbf{q})$, where $\mathbf{q} : \Omega \to \mathbb{R}^n$ is a linear function such that $\nabla\mathbf{q} = \mathbf{Q}$ and $\int_{2B} \mathbf{u} - \mathbf{q}\,dx = 0$. We use $\boldsymbol{\xi}$ as a test function for (3.1) and get

$$0 = \fint_{2B} \big(\mathbf{A}(\nabla\mathbf{u}) - \mathbf{A}(\mathbf{Q})\big) : \nabla\big(\eta^s(\mathbf{u} - \mathbf{q})\big)\,dx.$$

Using $\nabla(\eta^s(\mathbf{u}-\mathbf{q})) = \eta^s(\nabla\mathbf{u}-\mathbf{Q}) + s\eta^{s-1}\nabla\eta\otimes(\mathbf{u}-\mathbf{q})$ (2.13d), and $|\nabla\eta| \leq c/R$ we get

$$\fint_{2B} \eta^s\big(\mathbf{A}(\nabla\mathbf{u}) - \mathbf{A}(\mathbf{Q})\big) : (\nabla\mathbf{u} - \mathbf{Q})\,dx \leq c\fint_{2B} \eta^{s-1}\varphi'_{|\mathbf{Q}|}(|\nabla\mathbf{u} - \mathbf{Q}|)\,\frac{|\mathbf{u} - \mathbf{q}|}{R}\,dx$$

Using Lemma 2.4 and Young's inequality (2.2) we deduce

$$\begin{aligned}\fint_{2B} \eta^s|\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\,dx &\leq \delta\fint_{2B} (\varphi_{|\mathbf{Q}|})^*\Big(\eta^{s-1}\varphi'_{|\mathbf{Q}|}(|\nabla\mathbf{u} - \mathbf{Q}|)\Big)\,dx\\ &\quad + c_\delta\fint_{2B} \varphi_{|\mathbf{Q}|}\bigg(\frac{|\mathbf{u} - \mathbf{q}|}{R}\bigg)\,dx.\end{aligned} \tag{3.7}$$

We estimate the first term on the right-hand side using (2.17), $(s-1)q_2 = s$ (2.4), and Lemma 2.4 by

$$\begin{aligned}\delta\fint_{2B} (\varphi_{|\mathbf{Q}|})^*\Big(\eta^{s-1}\varphi'_{|\mathbf{Q}|}(|\nabla\mathbf{u} - \mathbf{Q}|)\Big)\,dx &\leq \delta\,c\fint_{2B} \eta^s\varphi_{|\mathbf{Q}|}(|\nabla\mathbf{u} - \mathbf{Q}|)\,dx\\ &\leq \delta\,c\fint_{2B} \eta^s|\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\,dx.\end{aligned}$$

So we can absorb this term for small $\delta > 0$ in (3.7) on the left-hand side. The second term on the right-hand side of (3.7) can be estimated with the help of the

Poincaré inequality, (see Theorem 3.2), $⨍_{2B} \mathbf{u} - \mathbf{q}\, dx = 0$, and Lemma 2.4 by

$$c_\delta ⨍_{2B} \varphi_{|\mathbf{Q}|}\left(\frac{|\mathbf{u}-\mathbf{q}|}{R}\right) dx \le c_\delta\, c\left(⨍_{2B} \big((\varphi_{|\mathbf{Q}|})(|\nabla \mathbf{u} - \mathbf{Q}|)\big)^\theta\, dx\right)^{\frac{1}{\theta}}$$
$$\le c_\delta\, c\left(⨍_{2B} \big|\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{Q})\big|^{2\theta}\, dx\right)^{\frac{1}{\theta}}$$

where $\theta \in (0,1)$ is as in Theorem 3.2. Combining the estimates starting with (3.7) we have proved (3.6). □

Note that Lemma 3.4 is an improved version of Theorems 3.1 and 3.2. Indeed, if we combine Theorems 3.1 and 3.2 with the estimate $\varphi(|\nabla\mathbf{u}|) \sim |\mathbf{V}(\nabla\mathbf{u})|^2$ from Lemma 2.4, then we immediately get 3.5 with $\mathbf{Q} = \mathbf{0}$. Lemma 3.4 is an improvement, since we are allow to substract $\mathbf{V}(\mathbf{Q})$ in the integrals, providing us with a reverse Hölder estimate for the oscillation.

Applying the ingenious lemma of Giaquinta and Modica to Lemma 3.4 we immediately get the following improved version of Theorem 3.3.

Lemma 3.5. *Let φ satisfy Assumption 2.1 and let $\mathbf{u}$ be a local minimizer of the functional (1.1). Then there exists $q_3 > 1$ and $c > 1$ such that for all balls B with $2B \Subset \Omega$, all $q \in [1, q_3]$, and all $\mathbf{Q} \in \mathbb{R}^{N\times n}$ holds*

$$\left(⨍_B |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\mathbf{Q})|^{2q}\, dx\right)^{\frac{1}{q}} \le c ⨍_{2B} |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\, dx \tag{3.8}$$

Note that c and q_3 only depend on the constant in (2.7).

Let us introduce the following notations: For $x, s \in \mathbb{R}^n$ we define

$$T_s(x) := x + s, \quad (\tau_s f)(x) := f(x+s) - f(x).$$

The following result is based on [6, Theorem 11].

Theorem 3.6. *Let φ satisfy Assumption 2.1 and let $\mathbf{u}$ be a local minimizer of the functional (1.1). Then there exists $c > 0$ such that if $B \subset \Omega$ is a ball with $2B \Subset \Omega$ and if $h \in \mathbb{R}^n \setminus \{0\}$ with $|h| \le R$, where R is the radius of B, then*

$$⨍_B |\tau_h \mathbf{V}(\nabla\mathbf{u})|^2\, dx \le c\,\frac{|h|^2}{R^2} ⨍_{2B} |\mathbf{V}(\nabla\mathbf{u})|^2\, dx. \tag{3.9}$$

Passing in Theorem 3.6 to $h \to 0^+$ we immediately get the following estimate for the gradients of $\mathbf{V}(\nabla\mathbf{u})$.

Corollary 3.7. *Let φ satisfy Assumption 2.1 and let $\mathbf{u}$ be a local minimizer of the functional (1.1). Then $\mathbf{V}(\nabla\mathbf{u}) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and there exists $c > 0$ such that if $B \subset \Omega$ is a ball with radius R and $2B \Subset \Omega$, then*

$$⨍_B \big|\nabla\big(\mathbf{V}(\nabla\mathbf{u})\big)\big|^2\, dx \le \frac{c}{R^2} ⨍_{2B} \big|\mathbf{V}(\nabla\mathbf{u})\big|^2\, dx. \tag{3.10}$$

Corollary 3.8. *Let φ satisfy Assumption 2.1 and let $\mathbf{u}$ be a local minimizer of the functional (1.1). If $n \geq 3$, then there exists $c > 0$ such that if $B \subset \Omega$ is a ball with $2B \Subset \Omega$, then*

$$\left(\fint_B \left| \mathbf{V}(\nabla \mathbf{u}) \right|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \leq c \fint_{2B} |\mathbf{V}(\nabla \mathbf{u})|^2 \, dx. \tag{3.11}$$

If $n = 2$, then the inequality holds if we replace on the left-hand side $\frac{2n}{n-2}$ and $\frac{n-2}{2n}$ by q and $\frac{1}{q}$, respectively, where $q \in [1, \infty)$. In this case $c = c(q)$. If $n = 1$, then we can use $q = \infty$.

Proof. The result follows from the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ for $n \geq 3$, $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for $n = 2$ and $q \in [1, \infty)$, and $W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $n = 1$. □

4. Approximated system

In order to study system (1.1) it is sometimes more convenient to examine an approximated version of the system. For the approximation we use the shifted-N-functions as introduced in (2.12). We will see that φ_λ is a good approximation of φ while it has a better behaviour at zero, see in particular Lemma 2.6. Later we will pass to the limit $\lambda \to 0$ to transfer our results to the original system (1.1).

If not stated otherwise we will assume that φ satisfies Assumption 2.1. As an approximation of (1.1) we consider for $\lambda \geq 0$ the functional

$$\mathcal{F}_\lambda(\mathbf{v}) = \int_\Omega \varphi_\lambda(|\nabla \mathbf{v}|) \, dx. \tag{4.1}$$

In analogy to (2.11) we define $\mathbf{A}_\lambda, \mathbf{V}_\lambda : \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ for $\lambda \geq 0$ by

$$\mathbf{A}_\lambda(\mathbf{Q}) = \varphi'_\lambda(|\mathbf{Q}|) \frac{\mathbf{Q}}{|\mathbf{Q}|}, \tag{4.2a}$$

$$\mathbf{V}_\lambda(\mathbf{Q}) = \psi'_\lambda(|\mathbf{Q}|) \frac{\mathbf{Q}}{|\mathbf{Q}|}, \tag{4.2b}$$

where ψ_λ is the associated N-function (compare (2.9)) given by

$$\psi'_\lambda(t) := \sqrt{\varphi'_\lambda(t)\, t}\,. \tag{4.3}$$

Since

$$\psi'_\lambda(t) = \sqrt{\varphi'_\lambda(t)\, t} = \sqrt{\frac{\varphi'(\lambda + t)}{\lambda + t} t^2} = \frac{\psi'(\lambda + t)}{\lambda + t} t$$

for $\lambda, t > 0$, the function ψ_λ is just the shifted version of ψ. Therefore, there will be no confusion in the notation.

Since $\varphi_0 = \varphi$, we recover for $\lambda = 0$ in (4.1) our original system (1.1). In particular, $\mathcal{F}_0 = \mathcal{F}$, $\mathbf{A}_0 = \mathbf{A}$, $\mathbf{V}_0 = \mathbf{V}$, and $\psi_0 = \psi$.

Remark 4.1. Let φ be an N-function with $\Delta_2(\{\varphi,\varphi^*\}) < \infty$. The following version of Young's inequality has been shown in [6, Lemma 32]. It holds

$$\varphi_\lambda'(t)s \le c\varphi_\lambda(t) + c\varphi_\lambda(s) \tag{4.4}$$

for all $\lambda, s, t \ge 0$, where c does only depend on $\Delta_2(\{\varphi,\varphi^*\})$. It has been shown in [6, Lemma 30] that

$$\varphi_\lambda(\theta\,\lambda) \le c\,\theta^2\varphi(\lambda), \tag{4.5}$$

for all $\lambda \ge 0$ and $\theta \in [0,1]$, where c does only depend on $\Delta_2(\{\varphi,\varphi^*\})$. Moreover, it has been shown in [7, Lemma 25 + Corollary 26] that

$$\varphi_{|\mathbf{Q}|}(t) \le c\,\varphi_{|\mathbf{P}|}(t) + c\big|\mathbf{V}(\mathbf{Q}) - \mathbf{V}(\mathbf{P})\big|^2 \tag{4.6}$$

for all $\mathbf{Q},\mathbf{P} \in \mathbb{R}^{N\times n}$ and all $t \ge 0$ and

$$\varphi_\lambda(t) \le c\,\big(\varphi(t) + \varphi(\lambda)\big), \tag{4.7}$$

for all $t, \lambda \ge 0$, where c does only depend on $\Delta_2(\{\varphi,\varphi^*\})$.

In particular, (4.7) implies that $L^{\varphi}_{\text{loc}}(\Omega) = L^{\varphi_\lambda}_{\text{loc}}(\Omega)$ and $W^{1,\varphi}_{\text{loc}}(\Omega) = W^{1,\varphi_\lambda}_{\text{loc}}(\Omega)$ for all $\lambda \ge 0$. So for all local results we can still work within the scope of the spaces L^φ and $W^{1,\varphi}$.

Moreover, it has been shown in [6, Lemma 26] that

$$(\varphi_\lambda)^*(t) \sim (\varphi^*)_{\varphi'(\lambda)}(t), \tag{4.8a}$$

$$\big((\varphi_\lambda)^*\big)'(t) \sim (\varphi^*)'_{\varphi'(\lambda)}(t) \tag{4.8b}$$

uniformly in $\lambda, t \ge 0$.

Lemma 4.2. *Let φ be an N-function with $\Delta_2(\{\varphi,\varphi^*\}) < \infty$. Then there exists $s_0 > 1$ such that $L^\varphi(\Omega) \hookrightarrow L^{s_0}_{\text{loc}}(\Omega)$. Note that s_0 only depends on $\Delta_2(\varphi,\varphi^*)$.*

Proof. Since $\Delta_2(\varphi^*) < \infty$ it follows from [17, Lemma 1.2.2+1.2.3] that φ^θ is quasiconvex for some $1 - \frac{1}{n} < \theta < 1$, i.e. there exists an N-function ρ with $\varphi^\theta \sim \rho$ and $\Delta_2(\{\rho,\rho^*\}) < \infty$. It is important to remark that θ and $\Delta_2(\{\rho,\rho^*\})$ only depend on $\Delta_2(\{\varphi,\varphi^*\})$. Let $s := \frac{1}{\theta}$, then $s > 1$. For $t \le 1$ we have $t^s \le 1$. On the other hand for $t \ge 1$, we have by the quasiconvexity of φ^θ that $\varphi(t) = \big(\varphi^\theta(t)\big)^{\frac{1}{\theta}} \ge \big(c\,t\,\varphi^\theta(1)\big)^{\frac{1}{\theta}} = c\,t^s\,\varphi(1)$. Overall, we have $t^s \le 1 + c\,\varphi(t)/\varphi(1)$ for all $t \ge 0$. This proves the claim. □

Due to Lemma 2.6 the result above also holds with φ replaced by φ_λ, where s_0 is independent of $\lambda > 0$.

Lemma 4.3. *Let φ satisfy Assumption 2.1. Then there exists $s_1 > 1$ such that the following holds. If $\lambda > 0$ and $\mathbf{u}_\lambda$ is a local minimizer of the functional (4.1) then $\mathbf{u}_\lambda \in W^{2,s_1}_{\text{loc}}(\Omega)$.*

Proof. Let B be a ball with radius R and $2B \Subset \Omega$. We will show that $\mathbf{u}_\lambda \in W^{2,s}(B)$ for some $s > 1$, independent of B and $\lambda > 0$. Let $s_* > 1$ be such that $L^\varphi(\Omega) \hookrightarrow L^{s_*}(2B)$. Choose $s \in (1, 2)$ such that $\frac{s}{2-s} \le s_*$. Let $h \in \mathbb{R}^n \setminus \{0\}$ with $|h| \le R$. With Hölder's inequality with $\frac{2}{2-s}$ and $\frac{2}{s}$ we estimate

$$\begin{aligned}\big(|h|^{-1}|\tau_h \nabla \mathbf{u}_\lambda|\big)^s &\le \big(\varphi_\lambda''(|\nabla \mathbf{u}_\lambda| + |\tau_h \nabla \mathbf{u}_\lambda|)\big)^{\frac{-s}{2-s}} \\ &\quad + \varphi_\lambda''(|\nabla \mathbf{u}_\lambda| + |\tau_h \nabla \mathbf{u}_\lambda|)|h|^{-2}|\tau_h \nabla \mathbf{u}_\lambda|^2 \\ &=: (I) + (II).\end{aligned}$$

We will show that (I) and (II) are integrable over B with bounds independent of h. With Lemma 2.4 we estimate $(II) \le c\,|h|^{-2}|\tau_h \mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda)|^2$. So with Theorem 3.6 we see that (II) is integrable over B with bound independent of h. Let us now consider (I). For $t \ge 0$ we estimate with (2.7)

$$\frac{1}{\varphi_\lambda''(t)} \le c\,\frac{\lambda + t}{\varphi'(\lambda + t)} \le \frac{\lambda + t}{\varphi'(\lambda)}.$$

If we replace t by $|\nabla \mathbf{u}_\lambda| + |\tau_h \nabla \mathbf{u}_\lambda|$, integrate over B and use $L^\varphi(\Omega) \hookrightarrow L^{s_*}(2B) \hookrightarrow L^{\frac{s}{2-s}}(2B)$, then we see that (I) is integrable over B with bound independent of h. We have shown that

$$\int_B \big(|h|^{-1}|\tau_h \nabla \mathbf{u}_\lambda|\big)^s\,dx \le c,$$

where c depends on λ and $\mathbf{u}_\lambda$ but is independent of h. It follows that $\mathbf{u}_\lambda \in W^{2,s}(B)$. □

Lemma 4.4. *Let φ satisfy Assumption 2.1 and be C^2 on $[0, \infty)$. Let $B \subset \Omega$ be a ball with radius R and $2B \Subset \Omega$ and let $\mathbf{w} \in W^{2,s}(2B)$ for some $s > 1$ such that $\mathbf{V}(\nabla \mathbf{w}) \in W^{1,2}(2B)$. Then*

$$\varphi''(|\nabla \mathbf{w}|)|\partial_i \nabla \mathbf{w}|^2 \sim |\partial_i \mathbf{V}(\nabla \mathbf{w})|^2$$

almost everywhere with $i = 1, \ldots, n$. In particular,

$$\int_B \varphi''(|\nabla \mathbf{w}|)|\partial_i \nabla \mathbf{w}|^2\,dx \sim \int_B |\partial_i \mathbf{V}(\nabla \mathbf{w})|^2 dx.$$

Proof. Let $r \in (0, R)$. Then by Lemma 2.4 (applied to $\mathbf{V}$ and ψ) and $\psi''(t) \sim \sqrt{\varphi''(t)}$ it follows that

$$\varphi''(|\nabla \mathbf{w}| + |\tau_{re_i} \nabla \mathbf{w}|)r^{-2}|\tau_{re_i} \nabla \mathbf{w}|^2 \sim r^{-2}|\tau_{re_i} \mathbf{V}(\nabla \mathbf{w})|^2. \tag{4.9}$$

Since $\mathbf{V} \in W^{1,2}(B)$, we have $r^{-1}\tau_{re_i}\mathbf{V}(\nabla \mathbf{w}) \to \partial_i \mathbf{V}(\nabla \mathbf{w})$ in $L^2(B)$ for $r \to 0$ and $r^{-2}|\tau_{re_i}\mathbf{V}(\nabla \mathbf{w})|^2 \to |\partial_i \mathbf{V}(\nabla \mathbf{w})|^2$ in $L^1(B)$. So the right hand side of (4.9) converges in $L^1(B)$ to $|\partial_i \mathbf{V}(\nabla \mathbf{w})|^2$. It remains to prove that

$$\varphi''(|\nabla \mathbf{w}| + |\tau_{re_i} \nabla \mathbf{w}|)r^{-2}|\tau_{re_i} \nabla \mathbf{w}|^2 \to \varphi''(|\nabla \mathbf{w}|)|\partial_i \nabla \mathbf{w}|^2 \qquad \text{a.e. in } B, \tag{4.10}$$

since then the theorem of dominated convergence proves the claim. Now, $\mathbf{w} \in W^{2,s}(2B)$, so for a subsequence $r_k \to 0$ we have $\tau_{r_k e_i} \nabla \mathbf{w} \to \mathbf{0}$ and $r_k^{-1} \tau_{r_k e_i} \nabla \mathbf{w} \to \partial_i \nabla \mathbf{w}$ almost everywhere in B. Now, the continuity of φ'' on $[0, \infty)$ proves the claim. □

Lemma 4.5. *Let φ satisfy Assumption 2.1. If $\lambda > 0$ and $\mathbf{u}_\lambda$ is a local minimizer of the functional (4.1), B is a ball with radius R and $2B \Subset \Omega$, and $\mathbf{V}(\nabla \mathbf{u}_\lambda) \in L^{2q}(2B)$ with $q \geq 1$, then $\varphi_\lambda(|\nabla \mathbf{u}_\lambda|) \in W^{1,\frac{2q}{1+q}}(B)$, $\varphi_\lambda'(|\nabla \mathbf{u}_\lambda|)|\nabla^2 \mathbf{u}_\lambda| \in L^{\frac{2q}{1+q}}(B)$ and*

$$\left(\fint_B \left| \nabla \varphi_\lambda(|\nabla \mathbf{u}_\lambda|) \right|^{\frac{2q}{1+q}} \right)^{\frac{1+q}{2q}} \leq c \left(\fint_{2B} \left(\varphi_\lambda'(|\nabla \mathbf{u}_\lambda|)\, |\nabla^2 \mathbf{u}_\lambda| \right)^{\frac{2q}{1+q}} \right)^{\frac{1+q}{2q}} \leq c\, R^{-1} \left(\fint_{2B} |\mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda)|^{2q}\, dx \right)^{\frac{1}{q}}.$$

Moreover,

$$\partial_j \varphi_\lambda(|\nabla \mathbf{u}_\lambda|) = \varphi_\lambda'(|\nabla \mathbf{u}_\lambda|) \partial_j |\nabla \mathbf{u}_\lambda|$$

for $j = 1, \ldots, n$.

Proof. Let $\lambda > 0$. Then by Lemma 4.3 we have $\mathbf{u}_\lambda \in W^{2,s_1}_{\mathrm{loc}}(2B)$ and therefore $|\nabla \mathbf{u}_\lambda| \in W^{1,s_1}_{\mathrm{loc}}(2B)$. So $|\nabla \mathbf{u}_\lambda|$ is absolutely continuous on almost every line (parallel to the coordinate axes). Since φ_λ is Lipschitz on $[0, \infty)$, we get that

$$\left| \nabla \varphi_\lambda(|\nabla \mathbf{u}_\lambda|) \right| \leq \varphi_\lambda'(|\nabla \mathbf{u}_\lambda|)|\nabla^2 \mathbf{u}_\lambda| \tag{4.11}$$

on almost every line. This, Young's inequality with $q+1$ and $\frac{q+1}{q}$, and (2.7) imply

$$\begin{aligned} \left| \nabla \varphi_\lambda(|\nabla \mathbf{u}_\lambda|) \right|^{\frac{2q}{1+q}} &\leq \left(\varphi_\lambda'(|\nabla \mathbf{u}_\lambda|)|\nabla^2 \mathbf{u}_\lambda| \right)^{\frac{2q}{1+q}} \\ &\leq c \left(\sqrt{\varphi_\lambda(|\nabla \mathbf{u}_\lambda|)} \sqrt{\varphi_\lambda''(|\nabla \mathbf{u}_\lambda|)}\, |\nabla^2 \mathbf{u}_\lambda| \right)^{\frac{2q}{1+q}} \\ &= c \left(t_0 \sqrt{\varphi_\lambda(|\nabla \mathbf{u}_\lambda|)}\; t_0^{-1} \sqrt{\varphi_\lambda''(|\nabla \mathbf{u}_\lambda|)}\, |\nabla^2 \mathbf{u}_\lambda| \right)^{\frac{2q}{1+q}} \\ &\leq c \left(\varphi_\lambda(|\nabla \mathbf{u}_\lambda|) \right)^q t_0^{2q} + c\, \varphi_\lambda''(|\nabla \mathbf{u}_\lambda|) \left| \nabla^2 \mathbf{u}_\lambda \right|^2 t_0^{-2} \end{aligned}$$

on almost every line, where $t_0 > 0$ will be chosen later. From Lemma 4.3 we know that $\mathbf{u}_\lambda \in W^{2,s_1}_{\mathrm{loc}}(2B)$. So with Lemma 4.4 and Corollary 3.7 we deduce that

$$\fint_B \varphi_\lambda''(|\nabla \mathbf{u}_\lambda|)|\nabla^2 \mathbf{u}_\lambda|^2\, dx \leq c \fint_B \left| \nabla \mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda) \right|^2 dx \leq c R^{-2} \fint_{2B} \left| \mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda) \right|^2 dx.$$

On the other hand with Lemma 2.4 we have

$$\fint_B \left(\varphi_\lambda(|\nabla \mathbf{u}_\lambda|) \right)^q dx \leq c \fint_B |\mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda)|^{2q}\, dx.$$

Overall, we have shown that

$$\begin{aligned}
&\fint_B \big|\nabla\varphi_\lambda(|\nabla\mathbf{u}_\lambda|)\big|^{\frac{2q}{1+q}} \\
&\quad\le t_0^{2q} c \fint_B |\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)|^{2q}\,dx + c\,t_0^{-2}R^{-2}\fint_{2B}\big|\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)\big|^2\,dx \\
&\quad\le t_0^{2q} c \fint_{2B} |\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)|^{2q}\,dx + ct_0^{-2}R^{-2}\bigg(\fint_{2B}\big|\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)\big|^{2q}\,dx\bigg)^{1/q}.
\end{aligned}$$

Minimizing over $t_0 > 0$ proves

$$\fint_B \big|\nabla\varphi_\lambda(|\nabla\mathbf{u}_\lambda|)\big|^{\frac{2q}{1+q}} \le c\,R^{-\frac{2q}{1+q}}\bigg(\fint_{2B}|\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)|^{2q}\,dx\bigg)^{\frac{2}{1+q}}.$$

This proves the claim. □

Note that Theorem 3.3 and Lemma 2.6 ensures that the requirements of Lemma 4.5 are always satisfied for some $q > 1$, where q is independent of $\lambda > 0$.

Corollary 4.6. *Let φ satisfy Assumption 2.1. Let $\lambda > 0$, let $\mathbf{u}_\lambda$ be a local minimizer of the functional (4.1), and let B be a ball with radius R and $2B \Subset \Omega$. For $n \ge 3$ we have $\varphi_\lambda(|\nabla\mathbf{u}_\lambda|) \in W^{1,\frac{n}{n-1}}(B)$ and*

$$\bigg(\fint_B \big|\nabla\varphi_\lambda(|\nabla\mathbf{u}_\lambda|)\big|^{\frac{n}{n-1}}\bigg)^{\frac{n-1}{n}} \le c\,R^{-1}\fint_{2B}|\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)|^2\,dx. \tag{4.12}$$

The constant does not depend on $\lambda > 0$. If $n = 1$ or $n = 2$, then for all $s \in [1, 2)$ holds $\varphi_\lambda(|\nabla\mathbf{u}_\lambda|) \in W^{1,s}(B)$ and

$$\bigg(\fint_B \big|\nabla\varphi_\lambda(|\nabla\mathbf{u}_\lambda|)\big|^{s}\bigg)^{\frac{1}{s}} \le c\,R^{-1}\fint_{2B}|\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)|^2\,dx.$$

The constant does not depend on $\lambda > 0$.

Proof. The result immediately follows from Lemma 4.5, Corollary 3.7, and Corollary 3.8. □

We need the following auxiliary results.

Lemma 4.7. *Let φ satisfy Assumption 2.1. Further, let $a > 0$ and $U \subset [0, a] \times \mathbb{R}^{N\times n}$. Then $\sup_{(\lambda,\mathbf{Q})\in U}|\mathbf{Q}| < \infty$ if and only if $\sup_{(\lambda,\mathbf{Q})\in U}|\mathbf{A}_\lambda(\mathbf{Q})| < \infty$. Moreover, $\sup_{(\lambda,\mathbf{Q})\in U}|\mathbf{Q}| < \infty$ if and only if $\sup_{(\lambda,\mathbf{Q})\in U}|\mathbf{V}_\lambda(\mathbf{Q})| < \infty$.*

Proof. It suffices to prove the result for $\mathbf{A}_\lambda$, since the result for $\mathbf{V}_\lambda$ is the same with φ replaced by ψ. Let $\sup_{(\lambda,\mathbf{Q})\in U}|\mathbf{Q}|<\infty$, then by (4.7)

$$\sup_{(\lambda,\mathbf{Q})\in U}|\mathbf{A}_\lambda(\mathbf{Q})| = \sup_{(\lambda,\mathbf{Q})\in U}\varphi'_\lambda(|\mathbf{Q}|) \le \sup_{(\lambda,\mathbf{Q})\in U} c\left(\varphi'(|\mathbf{Q}|)+\varphi'(\lambda)\right) < \infty.$$

Assume now that $\sup_{(\lambda,\mathbf{Q})\in U}|\mathbf{A}_\lambda(\mathbf{Q})|<\infty$. With $(\mathbf{A}_\lambda^{-1})(\mathbf{Q}) = ((\varphi_\lambda)^*)'(|\mathbf{Q}|)\frac{\mathbf{Q}}{|\mathbf{Q}|}$, (4.8), and (4.7) (applied to φ^*) it follows that

$$\begin{aligned}\sup_{(\lambda,\mathbf{Q})\in U}|\mathbf{Q}| = \sup_{(\lambda,\mathbf{Q})\in U}|\mathbf{A}_\lambda^{-1}(\mathbf{A}_\lambda(\mathbf{Q}))| &= \sup_{(\lambda,\mathbf{Q})\in U}((\varphi_\lambda)^*)'(|\mathbf{A}_\lambda(\mathbf{Q})|)\\ &\le c\sup_{(\lambda,\mathbf{Q})\in U}(\varphi^*)'_{\varphi'(\lambda)}(|\mathbf{A}_\lambda(\mathbf{Q})|)\\ &\le c\sup_{(\lambda,\mathbf{Q})\in U}\Big((\varphi^*)'(|\mathbf{A}_\lambda(\mathbf{Q})|)+(\varphi^*)'\big(\varphi'(\lambda)\big)\Big)\\ &= c\sup_{(\lambda,\mathbf{Q})\in U}\Big((\varphi^*)'(|\mathbf{A}_\lambda(\mathbf{Q})|)+\lambda\Big) < \infty.\end{aligned}$$

This proves the assertion. □

Lemma 4.8. *Let φ satisfy Assumption 2.1. Then the $(\lambda,\mathbf{Q})\mapsto\mathbf{A}_\lambda(\mathbf{Q})$, $(\lambda,\mathbf{Q})\mapsto\mathbf{A}_\lambda^{-1}(\mathbf{Q})$, $(\lambda,\mathbf{Q})\mapsto\mathbf{V}_\lambda(\mathbf{Q})$, and $(\lambda,\mathbf{Q})\mapsto\mathbf{V}_\lambda^{-1}(\mathbf{Q})$ are continuous on $[0,\infty)\times\mathbb{R}^{N\times n}$.*

Proof. Let $(\lambda_k,\mathbf{Q}_k)\to(\lambda,\mathbf{Q})$ for $k\to\infty$. If $\lambda+|\mathbf{Q}|>0$, then

$$\mathbf{A}_{\lambda_k}(\mathbf{Q}_k) = \frac{\varphi'(\lambda_k+|\mathbf{Q}_k|)}{\lambda_k+|\mathbf{Q}_k|}\mathbf{Q}_k \to \frac{\varphi'(\lambda+|\mathbf{Q}|)}{\lambda+|\mathbf{Q}|}\mathbf{Q} = \mathbf{A}_\lambda(\mathbf{Q}).$$

If $\lambda+|\mathbf{Q}|=0$, then $\lambda=0$, $\mathbf{Q}=\mathbf{0}$, and $\mathbf{A}_0(\mathbf{Q})=\mathbf{0}$. Moreover,

$$\left|\mathbf{A}_{\lambda_k}(\mathbf{Q}_k)\right| = \frac{\varphi'(\lambda_k+|\mathbf{Q}_k|)}{\lambda_k+|\mathbf{Q}_k|}|\mathbf{Q}_k| \le \varphi'(\lambda_k+|\mathbf{Q}_k|)\to 0.$$

So $\mathbf{A}_{\lambda_k}(\mathbf{Q}_k)\to\mathbf{A}_\lambda(\mathbf{Q})$ also in this case. This proves that $(\lambda,\mathbf{Q})\mapsto\mathbf{A}_\lambda(\mathbf{Q})$ is continuous.

We will now show the continuity of $(\lambda,\mathbf{Q})\mapsto\mathbf{A}_\lambda^{-1}(\mathbf{Q})$. Let $(\lambda_k,\mathbf{Q}_k)\to(\lambda,\mathbf{Q})$. We set $\mathbf{P}_k:=\mathbf{A}_{\lambda_k}^{-1}(\mathbf{Q}_k)$ and $\mathbf{P}:=\mathbf{A}_\lambda^{-1}(\mathbf{Q})$. We have to show $\mathbf{P}_k\to\mathbf{P}$. By Lemma 4.7 it follows that $\mathbf{P}_k$ is bounded. Thus there exists a subsequence $\mathbf{P}_{k_j}$ that convergences to some $\overline{\mathbf{P}}\in\mathbb{R}^{N\times n}$. Now the continuity of $(\lambda,\mathbf{Q})\mapsto\mathbf{A}_\lambda(\mathbf{Q})$ implies $\mathbf{Q}_{k_j}=\mathbf{A}_{\lambda_{k_j}}(\mathbf{P}_{k_j})\to\mathbf{A}_\lambda(\overline{\mathbf{P}})$ for $j\to\infty$. Since $\mathbf{Q}_k\to\mathbf{Q}$ it follows that $\mathbf{A}_\lambda(\mathbf{P})=\mathbf{Q}=\mathbf{A}_\lambda(\overline{\mathbf{P}})$. This implies $\mathbf{P}=\overline{\mathbf{P}}$ and therefore $\mathbf{P}_{k_j}\to\mathbf{P}$. Since the argument works for any subsequence of $\mathbf{P}_k$ it follows that the whole sequence $\mathbf{P}_k$ converges to $\mathbf{P}$. This proves the assertion.

The claim for $\mathbf{V}_\lambda$ and $\mathbf{V}_\lambda^{-1}$ follows by replacing φ by ψ and the remark after (4.3). □

In the following we will show how to transfer results for the approximated system (4.1) back to our original system (1.1). We will do it only for Corollary 4.6.

Theorem 4.9. *Let φ satisfy Assumption 2.1 and let $\mathbf{u}$ be a local minimizer of the functional (1.1). Then Corollary 4.6 holds with φ_λ replaced by φ.*

Proof. We consider the case $n \geq 3$. The other case follows analogously. By a simple covering argument, it suffices to prove the result for balls B with $4B \Subset \Omega$. For $\lambda > 0$ let $\mathbf{u}_\lambda$ be the minimizer of

$$\mathcal{F}_\lambda(\mathbf{w}) = \int_{4B} \varphi_\lambda(|\nabla \mathbf{w}|)\,dx \tag{4.13}$$

within the class $\{\mathbf{w} \in W^{1,\varphi}(4B) : \mathbf{w} = \mathbf{u} \text{ on } \partial(4B)\}$. Using the properties of φ it is standard to see that such a minimizer exists and is unique. We will show that $\mathbf{u}_\lambda$ converges for $\lambda \to 0$ to $\mathbf{u}$. We proceed similarly as in [2].

By (4.7) and Corollary 3.7 we know $\mathbf{u}_\lambda$ is equibounded in $W^{1,\varphi}(2B)$ and $\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)$ is equibounded in $W^{1,2}(2B)$ for $\lambda \to 0$. So we have (passing to a subsequence)

$$\begin{aligned} \mathbf{u}_\lambda &\rightharpoonup \mathbf{v} && \text{in } W^{1,\varphi}(2B),\\ \mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda) &\rightharpoonup \mathbf{H} && \text{in } W^{1,2}(2B),\\ \mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda) &\to \mathbf{H} && \text{in } L^2(2B) \end{aligned}$$

for some $\mathbf{v} \in W^{1,\varphi}(2B)$ and $\mathbf{H} \in W^{1,2}(2B)$. Passing to a subsequence we have

$$\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda) \to \mathbf{H} \qquad \text{almost everywhere in } 2B.$$

So by Lemma 4.8 we deduce that $\nabla\mathbf{u}_\lambda \to \mathbf{V}^{-1}(\mathbf{H})$ almost everywhere in $2B$. Since the weak limit coincides with the pointwise limit, we get $\mathbf{V}^{-1}(\mathbf{H}) = \nabla\mathbf{v}$ and $\mathbf{H} = \mathbf{V}(\nabla\mathbf{v})$. Since by Corollary 4.6 $\varphi_\lambda(|\nabla\mathbf{u}_\lambda)|$ is equibounded in $W^{1,\frac{n}{n-1}}(2B)$, there holds $\varphi_\lambda(|\nabla\mathbf{u}_\lambda|) \rightharpoonup g$ in $W^{1,\frac{n}{n-1}}(2B)$ for a subsequence. Since $\nabla\mathbf{u}_\lambda \to \nabla\mathbf{v}$ almost everywhere, we get $\varphi_\lambda(|\nabla\mathbf{u}_\lambda|) \rightharpoonup \varphi(|\nabla\mathbf{v}|)$ in $W^{1,\frac{n}{n-1}}(2B)$. This is enough to pass in (4.12) (for our subsequence) to the limit $\lambda \to 0$ and we see that (4.12) holds with φ_λ and $\mathbf{u}_\lambda$ replaced by φ and $\mathbf{v}$, respectively. The uniqueness of the minimizer of (4.13) for $\lambda = 0$ implies $\mathbf{u} = \mathbf{v}$. □

5. Subsolution

As in the paper by Uhlenbeck [26] and Acerbi and Fusco [1] we prove that the nonlinear quantity $\varphi(|\nabla\mathbf{u}|)$ is a subsolution of an uniformly elliptic system. As a first step, we will show this for the approximated system (4.1). We proceed similarly to [1].

Lemma 5.1. *Let φ be an N-function with $\Delta_2(\varphi) < \infty$. Then there exists $s > 1$ such that*

$$\left(\frac{\varphi(t)}{t}\right)^s \leq 2\left(\varphi(1)\right)^{s-1}\varphi(t) + \left(\varphi(1)\right)^s \tag{5.1}$$

uniformly for all $t > 0$. Note that $s > 1$ only depends on $\Delta_2(\varphi)$.

Proof. Let $K \geq 2$ denote the Δ_2-constant of φ. Then there exists $k \in \mathbb{N}$ with $K \leq 2^k$. Define $s := 1 + \frac{1}{k}$.

If $t \in (0, 1]$, then by convexity $\varphi(t)/t \leq \varphi(1)$, which proves (5.1). Assume now, that $t > 1$. Choose $m \in \mathbb{N}$ such that $2^{m-1} < t \leq 2^m$, then

$$\begin{aligned}\big(\varphi(t)\big)^s &\leq \big(\varphi(t)\big)^{\frac{1}{k}}\varphi(t) \leq \big(\varphi(2^m)\big)^{\frac{1}{k}}\varphi(t) \leq K^{\frac{m}{k}}\big(\varphi(1)\big)^{\frac{1}{k}}\varphi(t)\\ &\leq 2^m\big(\varphi(1)\big)^{\frac{1}{k}}\varphi(t) \leq 2\,t\big(\varphi(1)\big)^{\frac{1}{k}}\varphi(t).\end{aligned}$$

This implies for $t > 1$ that

$$\left(\frac{\varphi(t)}{t}\right)^s \leq \varphi(t)2\,t^{1-s}\big(\varphi(1)\big)^{\frac{1}{k}} \leq 2\,\big(\varphi(1)\big)^{s-1}\varphi(t).$$

This proves the claim. □

Using (2.7), the convexity of φ, and $\Delta_2(\varphi) < \infty$ we have

$$\varphi_\lambda''(t) \leq c\,\varphi''(\lambda+t) \leq c\,\frac{\varphi(\lambda+t)}{(\lambda+t)^2} \leq c\,\lambda^{-2}\varphi(\lambda+t) \leq c\,\lambda^{-2}\big(\varphi(\lambda)+\varphi(t)\big) \tag{5.2}$$

uniformly in $\lambda > 0$ and $t \geq 0$.

Lemma 5.2. *Let φ satisfy Assumption 2.1. Then there exists $s_2 > 1$ such that if $\lambda > 0$ and $\mathbf{u}_\lambda$ is a local minimizer of the functional (4.1), then $\mathbf{A}_\lambda(\nabla\mathbf{u}_\lambda) \in W^{1,s_2}_{\mathrm{loc}}(\Omega)$.*

Proof. Choose $s > 1$ as in Lemma 5.1 and let $s_2 := 2s/(1+s) \in (1, 2)$, then $s = s_2/(2-s_2)$ and $1 < s_2 < s$. Let $B \subset \Omega$ be a ball with radius R such that $2B \Subset \Omega$. Then $\mathbf{u}_\lambda \in W^{1,\varphi_\lambda}(2B)$ and $\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda) \in W^{1,2}(2B)$. With Lemma 2.4 and Lemma 5.1 we estimate

$$\begin{aligned}|\mathbf{A}_\lambda(\nabla\mathbf{u}_\lambda)|^s &\leq c\big(\varphi_\lambda'(|\nabla\mathbf{u}_\lambda|)\big)^s \leq c\left(\frac{\varphi_\lambda(|\nabla\mathbf{u}_\lambda|)}{|\nabla\mathbf{u}_\lambda|}\right)^s\\ &\leq 2\big(\varphi_\lambda(1)\big)^{s-1}\varphi_\lambda(|\nabla\mathbf{u}_\lambda)|) + c\,\big(\varphi_\lambda(1)\big)^s.\end{aligned}$$

Since $\mathbf{u}_\lambda \in W^{1,\varphi_\lambda}(B)$, it follows from the estimate above that $\mathbf{A}_\lambda(\nabla\mathbf{u}_\lambda) \in L^s(B)$. Since $s_2 < s$, we also get $\mathbf{A}_\lambda(\nabla\mathbf{u}_\lambda) \in L^{s_2}(B)$.

Let $h \in \mathbb{R}^n \setminus \{0\}$ with $|h| < R$. Then

$$\int_B |h|^{-2}\big|\tau_h\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)\big|^2\,dx \leq \int_{2B}\big|\nabla\big(\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)\big)\big|^2\,dx. \tag{5.3}$$

With Lemma (2.4) and Young's inequality, with $q = 2/s_2$ and $q' = 2/(2-s_2)$, we estimate

$$\begin{aligned}\big(|h|^{-1}|\tau_h\mathbf{A}_\lambda(\nabla\mathbf{u}_\lambda)|\big)^{s_2} &\leq c\,\big(\varphi_\lambda''(|\nabla\mathbf{u}_\lambda| + |\tau_h\nabla\mathbf{u}_\lambda|)|h|^{-1}|\tau_h\nabla\mathbf{u}_\lambda|\big)^{s_2}\\ &\leq \big(\varphi_\lambda''(|\nabla\mathbf{u}_\lambda| + |\tau_h\nabla\mathbf{u}_\lambda|)\big)^s\\ &\quad + \varphi_\lambda''(|\nabla\mathbf{u}_\lambda| + |\tau_h\nabla\mathbf{u}_\lambda|)|h|^{-2}|\tau_h\nabla\mathbf{u}_\lambda|^2\\ &\leq \big(\varphi_\lambda''(|\nabla\mathbf{u}_\lambda| + |\tau_h\nabla\mathbf{u}_\lambda|)\big)^s + c\,|h|^{-2}|\tau_h\mathbf{V}(\nabla\mathbf{u}_\lambda)|^2\\ &=: (I) + (II).\end{aligned}$$

Due to (5.3) the term (II) is in $L^1(B)$ with bound independently of h. With $\varphi''_\lambda(t) \sim \varphi(\lambda+t)/(\lambda+t)^2$ we estimate (I) by Lemma 5.1 as follows:

$$
\begin{aligned}
(I) &\le c\left(\frac{\varphi(\lambda+|\nabla \mathbf{u}_\lambda|+|\tau_h\nabla \mathbf{u}_\lambda|)}{(\lambda+|\nabla \mathbf{u}_\lambda|+|\tau_h\nabla \mathbf{u}_\lambda|)^2}\right)^s \\
&\le c\,\lambda^{-s}\left(\frac{\varphi(\lambda+|\nabla \mathbf{u}_\lambda|+|\tau_h\nabla \mathbf{u}_\lambda|)}{(\lambda+|\nabla \mathbf{u}_\lambda|+|\tau_h\nabla \mathbf{u}_\lambda|)}\right)^s \\
&\le c\big(\varphi(1),s\big)\,\lambda^{-s}\varphi(\lambda+|\nabla \mathbf{u}_\lambda|+|\tau_h\nabla \mathbf{u}_\lambda|)+c(\varphi(1),s)\lambda^{-s}.
\end{aligned}
$$

The convexity of φ and $\Delta_2(\varphi)<\infty$ imply that

$$
(I)\le c(\varphi(1),s)\,\lambda^{-s}\big(\varphi(\lambda)+\varphi(|\nabla \mathbf{u}_\lambda|)+\varphi(|\nabla \mathbf{u}_\lambda(h+\cdot)|)+c(\varphi(1),s)\lambda^{-s}.
$$

Since $\mathbf{u}_\lambda \in W^{1,\varphi}(2B)$, we get that (I) is in $L^1(B)$ with bound independently of h.

Overall, we have shown that $|h|^{-1}\tau_h \mathbf{A}_\lambda(\nabla \mathbf{u}_\lambda)$ is in $L^s(B)$ with bound independent of h. Thus $\mathbf{A}_\lambda(\nabla \mathbf{u}_\lambda)\in W^{1,s}(B)$. □

Lemma 5.3. *Let φ satisfy Assumption 2.1 and let $\lambda>0$. Let $\mathbf{G}:\mathbb{R}^n\to\mathbb{R}^{N\times n}$ satisfy $\mathbf{G}\in W^{1,s}(B)$ and $\mathbf{A}_\lambda(\mathbf{G})\in W^{1,s}(B)$ for some $s>1$. Then almost everywhere on B*

$$
\partial_i A_\lambda^{jk}(\mathbf{G})=\frac{\varphi'_\lambda(|\mathbf{G}|)}{|\mathbf{G}|}\partial_i G_{jk}+\left(\varphi''_\lambda(|\mathbf{G}|)-\frac{\varphi'_\lambda(|\mathbf{G}|)}{|\mathbf{G}|}\right)\frac{G_{jk}}{|\mathbf{G}|}\partial_i|\mathbf{G}|. \tag{5.4}
$$

Proof. Since $\mathbf{G}$ and $\mathbf{A}_\lambda(\mathbf{G})$ are in $W^{1,s}(B)$, they are absolutely continuous on almost every line (parallel to the coordinate axes). Due to Lemma 2.6, we know that $\mathbf{Q}\mapsto A_\lambda(\mathbf{Q})$ is C^1 on $\mathbb{R}^{N\times n}$. So (5.4) follows immediately on almost every line. □

Let φ satisfy Assumption 2.1. For $\lambda>0$ and $t\ge 0$ we define

$$
\omega_\lambda(t):=\frac{\varphi''_\lambda(t)\,t-\varphi'_\lambda(t)}{\varphi'_\lambda(t)}. \tag{5.5}
$$

Then by (2.7) and (2.14) it follows that there exists $c_0, c_1>0$ such that

$$
c_0-1\le\omega_\lambda(t)\le c_1. \tag{5.6}
$$

for all $t\ge 0$ and all $\lambda>0$. Note that c_0 and c_1 only depend on the constant in (2.7).

The next lemma shows that $\varphi_\lambda(|\nabla \mathbf{u}_\lambda|)$ is a subsolution to a uniformly elliptic problem, where the constants of ellipticity do no depend on $\lambda>0$.

Lemma 5.4. *Let φ satisfy Assumption 2.1. Let $\lambda>0$, let $\mathbf{u}_\lambda$ be a local minimizer of the functional (4.1), and let B be a ball with $2B\Subset\Omega$. Then there exists $\mathbf{G}_\lambda:2B\to\mathbb{R}^{n\times n}$ which is uniformly elliptic and $c_3>0$ (which is independent of λ) such that*

$$
\int\sum_{kl}\Big[G_\lambda^{kl}(\nabla \mathbf{u}_\lambda)\partial_l\big(\varphi_\lambda(|\nabla \mathbf{u}_\lambda|)\big)\Big]\partial_k\eta\,dx\le -c\int\eta\big|\nabla \mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda)\big|^2dx\le 0 \tag{5.7}
$$

holds for all $\eta \in C_0^1(2B)$, $\eta \geq 0$. *Moreover,*

$$\min\{c_0, 1\}|\boldsymbol{\xi}|^2 \leq \sum_{k,l} G_\lambda^{kl}(\mathbf{Q})\xi_k\xi_l \leq (c_1+1)|\boldsymbol{\xi}|^2 \tag{5.8}$$

for all $\mathbf{Q} \in \mathbb{R}^{n\times N}$ *and all* $\boldsymbol{\xi} \in \mathbb{R}^n$, *where* $c_0, c_1 > 0$ *are the constants from* (5.6), *which depend on the constant in* (2.7) *but are independent of* $\lambda > 0$.

Proof. Let $\eta \in C_0^1(2B)$. Let B be a ball of radius R and let $h \in \mathbb{R}^n \setminus \{0\}$ with $|h| \leq \min\{\mathrm{dist}(\mathrm{supp}(\eta), \partial(2B)), 1\}$. Define $\boldsymbol{\xi} := |h|^{-2}\tau_{-h}(\eta\tau_h\mathbf{u}_\lambda)$, then $\boldsymbol{\xi} \in W_0^{1,\varphi}(2B)$, so $\boldsymbol{\xi}$ is an admissible test function for (3.1). This implies

$$\begin{aligned}
0 &= \int |h|^{-2}\sum_{j,k}\tau_h\big(A_\lambda^{jk}(\nabla\mathbf{u}_\lambda)\big)\partial_k(\eta\,\tau_h u_{\lambda,j})\,dx\\
&= \int |h|^{-2}\sum_{j,k}\tau_h\big(A_\lambda^{jk}(\nabla\mathbf{u}_\lambda)\big)(\partial_k\eta)\tau_h u_{\lambda,j}\,dx\\
&\quad+\int |h|^{-2}\sum_{j,k}\tau_h\big(A_\lambda^{jk}(\nabla\mathbf{u}_\lambda)\big)\eta\,\tau_h\partial_k u_{\lambda,j}\,dx\\
&=: (I)+(II).
\end{aligned}$$

By Lemma 2.4 there exists $c_3 > 0$ (independent of λ) such that

$$(II) \geq c_3\int \eta\,|h|^{-2}\big|\tau_h\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)\big|^2\,dx =: c_3\,(III). \tag{5.9}$$

We choose $h := re_l$ with $l \in \{1,\ldots,n\}$ and $0 < r \leq \mathrm{dist}(\mathrm{supp}(\eta), \partial(2B))$. Then with Corollary 3.7, we have for $r \to 0$

$$(III) \to \int \eta|\partial_l\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)|^2\,dx. \tag{5.10}$$

We claim that for $r \to 0$

$$(I) \to \int\sum_{j,k}\partial_l\big(A_\lambda^{jk}(\nabla\mathbf{u}_\lambda)\big)(\partial_k\eta)\partial_l u_{\lambda,j}\,dx \tag{5.11}$$

and the integral is well defined in L^1. By Lemma 5.2, we have $\mathbf{A}_\lambda(\nabla\mathbf{u}_\lambda) \in W^{1,s_2}(2B)$ for some $s_2 > 1$ and certainly we have $\mathbf{u}_\lambda \in W^{1,\varphi}(2B)$. So it follows (for a suitable subsequence) that

$$\begin{aligned}
(IV) &:= |h|^{-2}\sum_{j,k}\tau_h\big(A_\lambda^{jk}(\nabla\mathbf{u}_\lambda)\big)(\partial_k\eta)\tau_h u_{\lambda,j}\\
&\to \sum_{j,k}\partial_l\big(A_\lambda^{jk}(\nabla\mathbf{u}_\lambda)\big)(\partial_k\eta)\partial_l u_{\lambda,j} =: (V)
\end{aligned} \tag{5.12}$$

almost everywhere for $r \to 0$. So it remains to find a majorant for the left hand side of (5.12) that converges in $L^1(2B)$ in order to conclude (5.11) by the dominated convergence Theorem. For these calculations we can keep η fixed, so the constants in the following estimates for (IV) may depend on η. By Lemma 2.4 we have

$$\big|(IV)\big| \le c\,|h|^{-2}|\tau_h\big(\mathbf{A}_\lambda(\nabla\mathbf{u}_\lambda)\big)||\tau_h\mathbf{u}_\lambda| \le c|h|^{-2}(\varphi_\lambda)'_{|\nabla\mathbf{u}_\lambda|}(|\tau_h\nabla\mathbf{u}_\lambda|)\,|\tau_h\mathbf{u}_\lambda|.$$

Now we proceed exactly as in [6] (see therein the estimate of (I_2) in the proof of Lemma 12). Define $T_{\sigma e_l} : \mathbb{R}^n \to \mathbb{R}^n$ by $T_{\sigma e_l}(x) := x + \sigma e_l$. Then $\tau_h\mathbf{u}_\lambda = \int_0^r \partial_l\mathbf{u}_\lambda \circ T_{\sigma e_l}\,d\sigma$, so we estimate

$$\big|(IV)\big| \le c\;|h|^{-2}(\varphi_\lambda)'_{|\nabla\mathbf{u}_\lambda|}(|\tau_h\nabla\mathbf{u}_\lambda|)\,r \fint_0^r |\nabla\mathbf{u}_\lambda \circ T_{\sigma e_l}|\,d\sigma.$$

With Young's inequality (4.4) for $(\varphi_\lambda)_{|\nabla\mathbf{u}_\lambda|}$ and Jensen's inequality we get

$$\big|(IV)\big| \le c\,|h|^{-2}(\varphi_\lambda)_{|\nabla\mathbf{u}_\lambda|}(|\tau_h\nabla\mathbf{u}_\lambda|) + c\,|h|^{-2}\fint_0^r (\varphi_\lambda)_{|\nabla\mathbf{u}_\lambda|}(r\,|\nabla\mathbf{u}_\lambda \circ T_{\sigma e_l}|)\,d\sigma.$$

By (4.6), (4.5), and Lemma 2.4 we have

$$\begin{aligned}(\varphi_\lambda)_{|\nabla\mathbf{u}_\lambda|}(r\,|\nabla\mathbf{u}_\lambda \circ T_{\sigma e_l}|) &\le c\,(\varphi_\lambda)_{|\nabla\mathbf{u}_\lambda\circ T_{\sigma e_l}|}(r\,|\nabla\mathbf{u}_\lambda \circ T_{\sigma e_l}|) + c\,|\mathbf{V}|(\nabla\mathbf{u}_\lambda)\\ &\qquad - \mathbf{V}(\nabla\mathbf{u}_\lambda \circ T_{\sigma e_l})^2\\ &\le c\,r^2\,(\varphi_\lambda)(|\nabla\mathbf{u}_\lambda \circ T_{\sigma e_l}|) + c\,|\tau_{\sigma e_l}\mathbf{V}(\nabla\mathbf{u}_\lambda)|^2.\\ &\le c\,r^2\,|\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda \circ T_{\sigma e_l})|^2 + c\,|\tau_{\sigma e_l}\mathbf{V}(\nabla\mathbf{u}_\lambda)|^2.\end{aligned}$$

So with the previous estimate and $|h| = r$ we get

$$\begin{aligned}\big|(IV)\big| &\le c\,|h|^{-2}|\tau_h\mathbf{V}(\nabla\mathbf{u}_\lambda)|^2 + c\fint_0^{|h|} |\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda \circ T_{\sigma e_l})|^2\,d\sigma\\ &\quad + c\,|h|^{-2}\fint_0^{|h|} |\tau_{\sigma e_l}\mathbf{V}(\nabla\mathbf{u}_\lambda)|^2\,d\sigma\\ &\le c\,|h|^{-2}|\tau_h\mathbf{V}(\nabla\mathbf{u}_\lambda)|^2 + c\fint_0^{|h|} |\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda \circ T_{\sigma e_l})|^2\,d\sigma\\ &\quad + c\fint_0^{|h|} |\sigma|^{-2}|\tau_{\sigma e_l}\mathbf{V}(\nabla\mathbf{u}_\lambda)|^2\,d\sigma.\end{aligned}$$

From Corollary 3.7 it follows that $|h|^{-1}\tau_h\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)$ converges to $\partial_l\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)$ in $L^2(2B)$ for $|h| \to 0$. As a consequence $|h|^{-2}|\tau_h\mathbf{V}(\nabla\mathbf{u}_\lambda)|^2 \to |\partial_l\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)|^2$ in $L^1(2B)$ as $|h| \to 0$. We will show now that the second and the third term on the right-hand side converge to $|\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)|^2$ and $|\partial_l\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)|^2$ in $L^1(2B)$ for $|h| \to 0$, respectively. First, observe that $\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda \circ T_{\sigma e_l}) = \mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda) \circ T_{\sigma e_l} \to \mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)$

in $L^2(2B)$ for $\sigma \to 0$. Thus, $|\mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda \circ T_{\sigma e_l})|^2 \to |\mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda)|^2$ in $L^1(2B)$ for $\sigma \to 0$. Second, we have already shown that $|\sigma|^{-2}|\tau_\sigma \mathbf{V}(\nabla \mathbf{u}_\lambda)|^2 \to |\partial_l \mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda)|^2$ in $L^1(2B)$ as $\sigma \to 0$. So the second and the third term on the right-hand side both look like $⨍_0^{|h|} a_\sigma\, d\sigma$ with some function $a_\sigma \in L^1(2B)$ with $a_\sigma \to a$ in $L^1(2B)$. We claim that $⨍_0^{|h|} a_\sigma\, d\sigma \to a$ in $L^1(2B)$. Indeed, for arbitrary $\varepsilon > 0$ there exists $\sigma_0 > 0$ such that $\|a_\sigma - a\|_{L^1(2B)} < \varepsilon$ for all $\sigma \in (0, \sigma_0)$. Now, we estimate for $|h| \le \sigma_0$.

$$\Bigg\| ⨍_0^{|h|} a_\sigma\, d\sigma - a \Bigg\|_{L^1(2B)} = \Bigg\| ⨍_0^{|h|} a_\sigma - a\, d\sigma \Bigg\|_{L^1(2B)} \le ⨍_0^{|h|} \|a_\sigma - a\|_{L^1(2B)}\, d\sigma \le \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves $⨍_0^{|h|} a_\sigma\, d\sigma \to a$ in $L^1(2B)$ for $|h| \to 0$. As a consequence we have shown the desired convergence of the second and third term to $|\mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda)|^2$ and $|\partial_l \mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda)|^2$ in $L^1(2B)$ for $|h| \to 0$, respectively. In total we have found a majorant of (IV) that converges in $L^1(2B)$. This and the already mentioned almost everywhere convergence prove our claim (5.11).

Using $(I) = -(II)$, (5.9), (5.10), and (5.11) we get after summation over $l = 1, \dots, n$

$$\int \sum_{l,j,k} \partial_l \big(A_\lambda^{jk}(\nabla \mathbf{u}_\lambda)\big)(\partial_k \eta)\partial_l u_{\lambda,j}\, dx \le -c_3 \int \eta \left|\nabla\big(\mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda)\big)\right|^2 dx \le 0. \tag{5.13}$$

Now, with Lemmas 4.3, 5.2 and 5.3, we calculate almost everywhere

$$\begin{aligned}
&\sum_{jl} \Big(\partial_l \big(A_\lambda^{jk}(\nabla \mathbf{u}_\lambda)\big)\partial_l u_{\lambda,j}\Big) \\
&= \sum_{jl} \Bigg[\bigg(\frac{\varphi_\lambda''(|\nabla \mathbf{u}_\lambda|)}{|\nabla \mathbf{u}_\lambda|} - \frac{\varphi_\lambda'(|\nabla \mathbf{u}_\lambda|)}{|\nabla \mathbf{u}_\lambda|^2}\bigg)\partial_l |\nabla \mathbf{u}_\lambda|\, \partial_k u_{\lambda,j}\, \partial_l u_{\lambda,j} \\
&\qquad + \varphi_\lambda'(|\nabla \mathbf{u}_\lambda|)\frac{\partial_l \partial_k u_{\lambda,j}}{|\nabla \mathbf{u}_\lambda|}\,(\partial_l u_{\lambda,j})\Bigg] \\
&= \sum_{jl} \Bigg(\frac{\partial_k u_{\lambda,j}\, \partial_l u_{\lambda,j}}{|\nabla \mathbf{u}_\lambda|^2}\Big(\varphi_\lambda''(|\nabla \mathbf{u}_\lambda|)\,|\nabla \mathbf{u}_\lambda| - \varphi_\lambda'(|\nabla \mathbf{u}_\lambda|)\Big)\partial_l |\nabla \mathbf{u}_\lambda|\Bigg) \\
&\qquad + \varphi_\lambda'(|\nabla \mathbf{u}_\lambda|)\partial_k |\nabla \mathbf{u}_\lambda| \\
&= \sum_{jl} \Bigg(\frac{\partial_k u_{\lambda,j}\, \partial_l u_{\lambda,j}}{|\nabla \mathbf{u}_\lambda|^2}\,\omega_\lambda(|\nabla \mathbf{u}_\lambda|)\,\varphi_\lambda'(|\nabla \mathbf{u}_\lambda|)\partial_l |\nabla \mathbf{u}_\lambda|\Bigg) + \varphi_\lambda'(|\nabla \mathbf{u}_\lambda|)\partial_k |\nabla \mathbf{u}_\lambda| \\
&= \sum_{jl} \Bigg(\frac{\partial_k u_{\lambda,j}\, \partial_l u_{\lambda,j}}{|\nabla \mathbf{u}_\lambda|^2}\,\omega_\lambda(|\nabla \mathbf{u}_\lambda|)\,\partial_l\big(\varphi_\lambda(|\nabla \mathbf{u}_\lambda|)\big)\Bigg) + \partial_k\big(\varphi_\lambda(|\nabla \mathbf{u}_\lambda|)\big),
\end{aligned}$$

where $\omega_\lambda : [0, \infty) \to \mathbb{R}$ is given by (5.5). Define $\mathbf{G}_\lambda : \mathbb{R}^{N\times n} \to \mathbb{R}^{n\times n}$ by

$$G_\lambda^{kl}(\mathbf{Q}) := \delta_{k,l} + \frac{\sum_j (Q_{jk}\, Q_{jl})}{|\mathbf{Q}|^2}\,\omega_\lambda(|\mathbf{Q}|).$$

Then

$$\sum_{jl}\Big(\partial_l\big(A_\lambda^{jk}(\nabla \mathbf{u}_\lambda)\big)\partial_l u_{\lambda,j}\Big) = \sum_l G_\lambda^{kl}(\nabla \mathbf{u}_\lambda)\partial_l\big(\varphi_\lambda(|\nabla \mathbf{u}_\lambda|)\big).$$

This together with (5.13) implies

$$\int \sum_{kl}\Big[G_\lambda^{kl}(\nabla \mathbf{u}_\lambda)\partial_l\big(\varphi_\lambda(|\nabla \mathbf{u}_\lambda|)\big)\Big]\partial_k\eta\,dx \leq -c\int \eta\,|\nabla \mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda)|^2\,dx \leq 0.$$

This proves (5.7). For all $\mathbf{Q}\in\mathbb{R}^{N\times n}$ and all $\boldsymbol{\xi}\in\mathbb{R}^n$ holds

$$\sum_{k,l} G_\lambda^{kl}(\mathbf{Q})\xi_k\xi_l = |\boldsymbol{\xi}|^2 + \frac{|\mathbf{Q}\boldsymbol{\xi}|^2}{|\mathbf{Q}|^2}\,\omega_\lambda(|\mathbf{Q}|).$$

This implies

$$\sum_{k,l} G_\lambda^{kl}(\mathbf{Q})\xi_k\xi_l \leq |\boldsymbol{\xi}|^2 + c_1\,|\boldsymbol{\xi}|^2 = (c_1+1)\,|\boldsymbol{\xi}|^2,$$
$$\sum_{k,l} G_\lambda^{kl}(\mathbf{Q})\xi_k\xi_l \geq |\boldsymbol{\xi}|^2\big(1+\min\{0,c_0-1\}\big) = \min\{c_0,1\},$$

where c_0 and c_1 are the constants from (5.6).

Let us transfer this result to our original system by passing to the limit $\lambda\to 0$.

Theorem 5.5. *Let φ satisfy Assumption 2.1, let $\mathbf{u}$ be a local minimizer of the functional (1.1), and let B be a ball with $4B \Subset \Omega$. Then there exists $\mathbf{G}: 2B \to \mathbb{R}^{n\times n}$ which is uniformly elliptic, i.e.*

$$\min\{c_0,1\}|\boldsymbol{\xi}|^2 \leq \sum_{k,l} G^{kl}(\mathbf{Q})\xi_k\xi_l \leq (c_1+1)|\boldsymbol{\xi}|^2$$

for all $\mathbf{Q}\in\mathbb{R}^{n\times N}$ and all $\boldsymbol{\xi}\in\mathbb{R}^n$, where $c_0, c_1 > 0$ only depend on the constant in (2.7), such that

$$\int \sum_{kl}\Big[G^{kl}(\nabla \mathbf{u})\partial_l\big(\varphi(|\nabla \mathbf{u}|)\big)\Big]\partial_k\eta\,dx \leq -c\int \eta\,\big|\nabla \mathbf{V}(\nabla \mathbf{u})\big|^2\,dx \leq 0 \tag{5.14}$$

holds for all $\eta\in C_0^1(2B)$, $\eta\geq 0$.

Proof. We proceed as in the proof of Theorem 4.9. In particular, let $\mathbf{u}_\lambda$ be the minimizer of $\mathcal{F}_\lambda(\mathbf{w}) = \int_{4B}\varphi_\lambda(|\nabla \mathbf{w}|)\,dx$ within the class $\{\mathbf{w}\in W^{1,\varphi}(4B) : \mathbf{w} = \mathbf{u} \text{ on } \partial(4B)\}$. Again we restrict ourselves to the case $n\geq 3$. The other case follows analogously. From the proof of Theorem 4.9 we know that $\mathbf{V}_\lambda(\nabla \mathbf{u}_\lambda) \rightharpoonup \mathbf{V}(\nabla \mathbf{u})$ in $W^{1,2}(2B)$ and $\nabla\varphi_\lambda(|\nabla \mathbf{u}_\lambda|) \rightharpoonup \nabla\varphi(|\nabla \mathbf{u}|)$ in $L^{\frac{n}{n-1}}(2B)$.

For any $\lambda > 0$ there exists by Lemma 5.4 a function $\mathbf{G}_\lambda : 2B \to \mathbb{R}^{n\times n}$ such that (5.7) holds for all $\eta\in C_0^1(4B)$. These $\mathbf{G}_\lambda$ satisfy (5.8) and are therefore uniformly elliptic independent of the choice of λ. In particular, there exists a sequence

$\lambda_j \to 0$ such that $\mathbf{G}_{\lambda_j}$ converges almost everywhere to some $\mathbf{G} : 2B \to \mathbb{R}^{n\times n}$ which is also uniformly elliptic and satisfies (5.8). So for $\eta \in C_0^1(2B)$ the term $G^{kl}_{\lambda_j}(\nabla \mathbf{u}_\lambda)\partial_k \eta$ in (5.7) converges strongly in $L^n(2B)$ to $G^{kl}\partial_k\eta$ for $\lambda \to \infty$. This and $\nabla\varphi_\lambda(|\nabla\mathbf{u}_\lambda|) \rightharpoonup \nabla\varphi(|\nabla\mathbf{u}|)$ in $L^{\frac{n}{n-1}}(2B)$ proves the convergence of the first integral in (5.7) to the first integral in (5.14) for $j \to \infty$. The convergence of the second integral for $j \to \infty$ follows by the lower semicontinuity of the integral with respect to weak convergence in $L^2(2B)$ and $\nabla\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda) \rightharpoonup \nabla\mathbf{V}(\nabla\mathbf{u})$ in $L^2(2B)$. It is obvious that the limit is non-positive. □

Due to Theorem 5.5 we know that $\varphi(|\nabla\mathbf{u}|)$ is a subsolution of a uniformly elliptic equation. If $\varphi(|\nabla\mathbf{u}|) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, then we could apply Harnack's inequality to get local L^∞ estimates. Unfortunately, we only know so far that $\varphi(|\nabla\mathbf{u}|) \in W^{1,\frac{n}{n-1}}_{\mathrm{loc}}(\Omega)$, see Theorem 4.9. To overcome this difficulty, we need the following result due to Marcellini and Papi, proved under more general hypotheses.

Proposition 5.6. [22, Theorem A] *Let φ satisfy Assumption 2.1. Let $\lambda > 0$ and let $\mathbf{u}_\lambda$ be a local minimizer of the functional (4.1). Then $\mathbf{u}_\lambda \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.*

Note that we need Proposition 5.6 only to justify $\varphi_\lambda(|\nabla\mathbf{u}_\lambda|) \in W^{1,2}_{\mathrm{loc}}(\Omega)$. We do not use any qualitative estimates for $\mathbf{u}_\lambda$ in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

Lemma 5.7. *Let φ satisfy Assumption 2.1. Let $\lambda > 0$ and let $\mathbf{u}_\lambda$ be a local minimizer of the functional (4.1). Then $\mathbf{u}_\lambda \in W^{2,2}_{\mathrm{loc}}(\Omega)$ and $\varphi_\lambda(|\nabla\mathbf{u}_\lambda|) \in W^{1,2}_{\mathrm{loc}}(\Omega)$.*

Proof. Let B be a ball with $2B \Subset \Omega$. Due to Proposition 5.6 we have $\mathbf{u}_\lambda \in W^{1,\infty}(B)$. Let $M := \|\nabla\mathbf{u}_\lambda\|_{L^\infty(B)}$. Due to Lemma 4.3 and Corollary 4.6 we have $\mathbf{u}_\lambda \in W^{2,s_1}_{\mathrm{loc}}(\Omega)$ and $\varphi_\lambda(|\nabla\mathbf{u}_\lambda|) \in W^{1,s_3}_{\mathrm{loc}}(\Omega)$ for some $s_1, s_3 > 1$. So by Corollary 3.7 we have $\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda) \in W^{1,2}(B)$. So with Lemma 4.4 and $\varphi_\lambda''(t) \sim \varphi''(\lambda+t) \sim \varphi(\lambda+t)/(\lambda+t)^2$ uniformly in $\lambda > 0$ and $t \geq 0$ we deduce that

$$\infty > \int_B \varphi_\lambda''(|\nabla\mathbf{u}_\lambda|)|\nabla^2\mathbf{u}_\lambda|^2 \chi_{\{|\nabla\mathbf{u}_\lambda|\leq M\}}\,dx \geq c\,\frac{\varphi(\lambda)}{(\lambda+M)^2}\int_B |\nabla^2\mathbf{u}_\lambda|^2\,dx.$$

This proves $\nabla\mathbf{u}_\lambda \in W^{2,2}(B)$. With Lemma 4.5 we estimate

$$\begin{aligned}\int_B \big|\nabla\varphi_\lambda(|\nabla\mathbf{u}_\lambda|)\big|^2\,dx &\leq c\int_B \big(\varphi_\lambda'(|\nabla\mathbf{u}_\lambda|)\big)^2|\nabla^2\mathbf{u}_\lambda|^2\,dx\\ &\leq c\big(\varphi_\lambda'(M)\big)^2\int_B |\nabla^2\mathbf{u}_\lambda|^2\,dx < \infty.\end{aligned}$$

This proves $\varphi_\lambda(|\nabla\mathbf{u}_\lambda|) \in W^{1,2}(B)$. □

Using the previous lemma we can now provide the desired $W^{1,2}$-estimate for $\varphi(|\nabla\mathbf{u}|)$.

Lemma 5.8. *Let φ satisfy Assumption 2.1, let $\mathbf{u}$ be a local minimizer of the functional (1.1), and let B be a ball with $2B \Subset \Omega$. Then*

$$\sup_B \varphi(|\nabla \mathbf{u}|) \leq c \fint_{2B} \varphi(|\nabla \mathbf{u}|)\, dx, \tag{5.15}$$

$$\fint_B |\nabla \varphi(|\nabla \mathbf{u}|)|^2\, dx \leq c\, R^{-2} \fint_{2B} \big|\varphi(|\nabla \mathbf{u}|)\big|^2\, dx. \tag{5.16}$$

Proof. First, we prove the lemma with φ replaced by φ_λ. Due to Lemma 5.7 we can apply Harnack's inequality to (5.7) to get (5.15). The estimate (5.16) follows as usual with the test function $\eta := \kappa^2 \varphi_\lambda(|\nabla \mathbf{u}_\lambda|)$ in Lemma 5.4, where $\kappa \in C_0^\infty(2B)$ with $\chi_B \leq \kappa \leq \chi_{2B}$ and $\|\nabla \kappa\|_\infty \leq c/R$. Now, exactly as in the proof of Theorems 4.9 and 5.5 the claim follows by passing to the limit $\lambda \to 0$. □

We will now apply the theory of subsolutions to $\varphi(|\nabla \mathbf{u}|)$ to derive the weak Harnack inequality.

Theorem 5.9. *Let φ satisfy Assumption 2.1. Let $\mathbf{u}$ be a local minimizer of the functional (1.1), and let B be a ball such that $2B \Subset \Omega$. Then*

$$\Phi\Big(\mathbf{u}, \frac{1}{2}B\Big) \leq c\Big(\sup_B \varphi(|\nabla \mathbf{u}|) - \sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|)\Big) \tag{5.17}$$

where $\Phi(\mathbf{u}, B) := \fint_B |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_B|^2\, dx$ is the excess functional.

Proof. Due to Theorem 5.5 we know that $\varphi(|\nabla \mathbf{u}|)$ is a subsolution of the uniformly elliptic equation (5.14). Moreover, by Lemma 5.8 we know that $\varphi(|\nabla \mathbf{u}|) \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Therefore, we can proceed exactly as in [1] and [15, Proposition 3.1] and apply Harnack's inequality for subsolutions to get (5.17). □

6. Hölder continuity of the gradients

In this section we will prove that $\mathbf{V}(\nabla \mathbf{u})$ and $\nabla \mathbf{u}$ are Hölder continuous. For this we have to strengthen our requirements on φ. In particular, we assume that φ satisfies Assumption 2.2, which states that φ'' is Hölder continuous away from zero. We will use this property of φ in form of additional regularity of $\mathbf{A}$, which is expressed in the following lemma.

Lemma 6.1. *Let φ satisfy Assumption 2.2 and let $\beta > 0$ be as Assumption 2.2. Then there exists $c > 0$ such that for all $\mathbf{H}, \mathbf{Q} \in \mathbb{R}^{N \times n}$ with $|\mathbf{H}| < \frac{1}{2}|\mathbf{Q}|$ holds*

$$\big|\nabla_{N\times n}\mathbf{A}(\mathbf{Q}+\mathbf{H}) - \nabla_{N\times n}\mathbf{A}(\mathbf{Q})\big| ds \leq c\, \varphi''(|\mathbf{Q}|)\bigg(\frac{|\mathbf{H}|}{|\mathbf{Q}|}\bigg)^\beta,$$

where c depends on φ only via the constants in (2.7) and (2.8).

Proof. The claim follows immediately from (2.19), (2.8), and Remark 2.3. □

Following the ideas of Lemma 2.10 in [1] we prove the following excess decay estimate.

Lemma 6.2. *Let φ satisfy Assumption 2.2 and let* $\mathbf{u}$ *be a local minimizer of the functional (1.1). Then there exists $c > 1$ such that for every $\tau \in (0, 1)$ there exists $\varepsilon_0 > 0$ such that for every ball $B \Subset \Omega$ holds*

$$\Phi(\mathbf{u}, B) \leq \varepsilon_0 \sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|) \Longrightarrow \Phi(\mathbf{u}, \tau B) \leq c\, \tau^2\, \Phi(\mathbf{u}, B).$$

Note that c and ε_0 depend on φ only via the constant in (2.7).

Proof. We fix $\tau \in (0, 1)$ and will choose $\varepsilon_0 > 0$ later. Without loss of generality we can assume $\tau \leq \frac{1}{2}$. Let $B \Subset \Omega$ be a ball such that $\Phi(\mathbf{u}, B) \leq \varepsilon_0 \sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|)$. Let $\mathbf{Q} \in \mathbb{R}^{N\times n}$ be such that

$$\mathbf{V}(\mathbf{Q}) = \langle \mathbf{V}(\nabla \mathbf{u})\rangle_B$$

From the Lemma 5.8, Lemma 2.4, and the definition of $\Phi(\mathbf{u}, B)$ we get

$$\sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|) \leq c \fint_B \varphi(|\nabla \mathbf{u}|)\, dx \leq c \fint_B |\mathbf{V}(\nabla \mathbf{u})|^2\, dx \leq c\big(\Phi(\mathbf{u}, B) + |\mathbf{V}(\mathbf{Q})|^2\big).$$

So if $\varepsilon_0 < \frac{1}{2}c$ we can conclude

$$\sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|) \leq c|\mathbf{V}(\mathbf{Q})|^2 \leq c\varphi(|\mathbf{Q}|), \tag{6.1}$$

where we have also used Lemma 2.4.

If $\mathbf{Q} = \mathbf{0}$, then we get by (6.1) that $\nabla \mathbf{u} = \mathbf{0}$ on $\frac{1}{2}B$ and the conclusion of the lemma is immediate. So in the following we can assume that $\mathbf{Q} \neq \mathbf{0}$.

Since φ is strictly increasing, we deduce from (6.1) that

$$\sup_{\frac{1}{2}B} |\nabla \mathbf{u}| \leq c\, |\mathbf{Q}|, \tag{6.2}$$

where we have also used that $\Delta_2(\varphi) < \infty$. This proves $|\nabla \mathbf{u}| + |\mathbf{Q}| \sim |\mathbf{Q}|$ on $\frac{1}{2}B$, so with Lemma 2.4, $\varphi(t) \sim \varphi''(t)\, t^2$, and $\Delta_2(\varphi) < \infty$, we get

$$\begin{aligned} \Phi\Big(\mathbf{u}, \frac{1}{2}B\Big) &\sim \fint_{\frac{1}{2}B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\, dx \\ &\sim \fint_{\frac{1}{2}B} \varphi''(|\mathbf{Q}| + |\nabla \mathbf{u}|)|\nabla \mathbf{u} - \mathbf{Q}|^2\, dx \\ &\sim \fint_{\frac{1}{2}B} \varphi''(|\mathbf{Q}|)|\nabla \mathbf{u} - \mathbf{Q}|^2\, dx. \end{aligned} \tag{6.3}$$

We define $\mathbf{w} := \mathbf{u} - \mathbf{q}$, where $\mathbf{q} : \Omega \to \mathbb{R}^n$ is a linear function such that $\nabla \mathbf{q} = \mathbf{Q}$ and $\int_{\tau B} \mathbf{u} - \mathbf{q}\, dx = 0$. Note that (6.2) implies that

$$|\nabla \mathbf{w}| = |\nabla \mathbf{u} - \mathbf{Q}| \leq c\,|\mathbf{Q}| \quad \text{on } \frac{1}{2}B. \tag{6.4}$$

Let $\mathbf{v}$ be the unique solution in $\mathbf{w} + W_0^{1,2}(\frac{1}{4}B, \mathbb{R}^N)$ satisfying

$$\sum_{ij\alpha\gamma} \int_{\frac{1}{4}B} \partial_{j\gamma} A_{i\alpha}(\mathbf{Q}) \partial_\gamma v_j \partial_\alpha z_i\, dx = 0 \tag{6.5}$$

for all $\mathbf{z} \in W_0^{1,2}(\frac{1}{4}B, \mathbb{R}^N)$. Then by Remark 2.12 the system (6.5) is uniformly elliptic with lower and upper constants of ellipticity proportional to $\varphi''(|\mathbf{Q}|)$. Thus by the theory of elliptic systems we have (with constants independent of $\mathbf{Q}$)

$$\fint_{\tau B} |\nabla \mathbf{v} - \langle \nabla \mathbf{v} \rangle_{\tau B}|^2\, dx \leq c\,\tau^2 \fint_{\frac{1}{4}B} |\nabla \mathbf{v} - \langle \nabla \mathbf{v} \rangle_{\frac{1}{4}B}|^2\, dx \tag{6.6}$$

for all $\tau \in (0, \frac{1}{4}]$. From (3.2), $\nabla \mathbf{u} = \mathbf{w} + \mathbf{Q}$, and Taylor's formula we get

$$\begin{aligned} 0 &= \sum_{i\alpha} \fint_{\frac{1}{4}B} \big(A_{i\alpha}(\nabla \mathbf{u}) - A_{i\alpha}(\mathbf{Q})\big) \partial_\alpha z_i\, dx \\ &= \sum_{ij\alpha\gamma} \fint_{\frac{1}{4}B} \int_0^1 \partial_{j\gamma} A_{i\alpha}(\mathbf{Q} + s\nabla \mathbf{w}(x))\, ds\, \partial_\gamma w_j \partial_\alpha z_i\, dx \end{aligned}$$

for all $\mathbf{z} \in W_0^{1,2}(\frac{1}{4}B)$. In combination with (6.5) this implies

$$\begin{aligned} &\sum_{ij\alpha\gamma} \fint_{\frac{1}{4}B} \partial_{j\gamma} A_{i\alpha}(\mathbf{Q})(\partial_\gamma v_j - \partial_\gamma w_j) \partial_\alpha z_i\, dx \\ &= \sum_{ij\alpha\gamma} \fint_{\frac{1}{4}B} \int_0^1 \big[\partial_{j\gamma} A_{i\alpha}(\mathbf{Q} + s\nabla \mathbf{w}(x)) - \partial_{j\gamma} A_{i\alpha}(\mathbf{Q})\big] ds\, \partial_\gamma w_j \partial_\alpha z_i\, dx. \end{aligned} \tag{6.7}$$

Our goal is to estimate the right-hand side of (6.7). For fixed $x \in \frac{1}{4}B$

$$(I) := \sum_{ij\alpha\gamma} \int_0^1 \big|\partial_{j\gamma} A_{i\alpha}(\mathbf{Q} + s\nabla \mathbf{w}(x)) - \partial_{j\gamma} A_{i\alpha}(\mathbf{Q})\big| ds.$$

Let $\beta > 0$ be as in Assumption 2.2 and $q_3 > 1$ as in Lemma 3.5. Define $\beta_2 := \min\{\beta, q_3 - 1\}$. We claim that

$$(I) \leq c\,\varphi''(|\mathbf{Q}|)\bigg(\frac{|\nabla \mathbf{w}(x)|}{|\mathbf{Q}|}\bigg)^{\beta_2}. \tag{6.8}$$

Case $|\nabla \mathbf{w}(x)| < \frac{1}{2}|\mathbf{Q}|$: Then $s|\nabla \mathbf{w}(x)| < \frac{1}{2}|\mathbf{Q}|$ for all $s \in [0,1]$. So with Lemma 6.1 we estimate

$$(I) \leq c \int_0^1 \varphi''(|\mathbf{Q}|)\bigg(\frac{|\nabla \mathbf{w}(x)|}{|\mathbf{Q}|}\bigg)^{\beta_2} ds \leq c\,\varphi''(|\mathbf{Q}|)\bigg(\frac{|\nabla \mathbf{w}(x)|}{|\mathbf{Q}|}\bigg)^{\beta_2}.$$

Case $|\nabla \mathbf{w}(x)| \geq \frac{1}{2}|\mathbf{Q}|$: Then (6.4) implies $|\nabla \mathbf{w}(x)| \sim |\mathbf{Q}|$. We estimate

$$(I) \leq \sum_{ij\alpha\gamma} \int_0^1 \big|\partial_{j\gamma} A_{i\alpha}(\mathbf{Q} + s\nabla \mathbf{w}(x))\big| + \big|\partial_{j\gamma} A_{i\alpha}(\mathbf{Q})\big| ds.$$

So with (2.19), Lemma 2.7, (2.7), $\Delta_2(\varphi) < \infty$, and $|\nabla \mathbf{w}(x)| \sim |\mathbf{Q}|$ we get

$$(I) \leq c\,\big(\varphi''(|\mathbf{Q}| + |\nabla \mathbf{w}|) + \varphi''(|\mathbf{Q}|)\big) \leq \varphi''(|\mathbf{Q}|) \leq c\,\varphi''(|\mathbf{Q}|)\bigg(\frac{|\nabla \mathbf{w}|}{|\mathbf{Q}|}\bigg)^{\beta}.$$

So we have proved our claim (6.8) in both cases.

We choose $\mathbf{z} = \mathbf{v} - \mathbf{w}$ in (6.7) and with (6.8) we estimate

$$\fint_{\frac{1}{4}B} \varphi''(|\mathbf{Q}|)\,|\nabla \mathbf{v} - \nabla \mathbf{w}|^2\,dx \leq c \fint_{\frac{1}{4}B} \varphi''(|\mathbf{Q}|)\,|\mathbf{Q}|^{-\beta_2}|\nabla \mathbf{w}|^{1+\beta_2}|\nabla \mathbf{v} - \nabla \mathbf{w}|\,dx,$$

where we have used again that (6.5) is uniformly elliptic with lower and upper constants of ellipticity proportional to $\varphi''(|\mathbf{Q}|)$. It follows that

$$\fint_{\frac{1}{4}B} \varphi''(|\mathbf{Q}|)\,|\nabla \mathbf{v} - \nabla \mathbf{w}|^2\,dx \leq c\,\varphi''(|\mathbf{Q}|)\,|\mathbf{Q}|^{-2\beta_2} \fint_{\frac{1}{4}B} |\nabla \mathbf{w}|^{2+2\beta_2}\,dx.$$

Note that due to (6.2), $\nabla \mathbf{w} = \nabla \mathbf{u} - \mathbf{Q}$ on $\frac{1}{2}B$, and Lemma 2.4 we have

$$\varphi''(|\mathbf{Q}|)|\nabla \mathbf{w}|^2 \sim \varphi''(|\mathbf{Q}| + |\nabla \mathbf{u}|)|\nabla \mathbf{u} - \mathbf{Q}|^2 \sim |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2. \tag{6.9}$$

So with the previous estimate and $\varphi(t) \sim \varphi''(t)t^2$ (see (2.4) and (2.7)) we get

$$\fint_{\frac{1}{4}B} \varphi''(|\mathbf{Q}|)\,|\nabla \mathbf{v} - \nabla \mathbf{w}|^2\,dx \leq c\,\big(\varphi(|\mathbf{Q}|)\big)^{-\beta_2} \fint_{\frac{1}{4}B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{Q})|^{2(1+\beta_2)}\,dx.$$

Now with Jensen's inequality and Lemma 3.5 (using $1 \le 1+\beta_2 \le q_3$) we deduce

$$\begin{aligned} \fint_{\frac{1}{4}B} \varphi''(|\mathbf{Q}|)\,|\nabla \mathbf{v} - \nabla \mathbf{w}|^2\,dx &\le c\,\big(\varphi(|\mathbf{Q}|)\big)^{-\beta_2} \fint_{\frac{1}{4}B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{Q})|^{2(1+\beta_2)}\,dx \\ &\le c\,\big(\varphi(|\mathbf{Q}|)\big)^{-\beta_2} \bigg(\fint_{\frac{1}{2}B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\,dx \bigg)^{1+\beta_2}. \end{aligned} \tag{6.10}$$

We will use this estimate later in our calculations. Let us now start to estimate $\Phi(\mathbf{u}, \tau B)$. Using Lemma 2.4 we estimate

$$\begin{aligned} \Phi(\mathbf{u}, \tau B) &= \fint_{\tau B} |\mathbf{V}(\nabla \mathbf{u}) - \langle \mathbf{V}(\nabla \mathbf{u})\rangle_{\tau B}|^2\,dx \\ &= \inf_{\mathbf{H} \in \mathbb{R}^{N\times N}} \fint_{\tau B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{H}|^2\,dx \\ &\le \fint_{\tau B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(\langle \nabla \mathbf{u}\rangle_{\tau B})|^2\,dx \\ &\le c \fint_{\tau B} \varphi''(|\nabla \mathbf{u}| + |\langle \nabla \mathbf{u}\rangle_{\tau B}|)\,|\nabla \mathbf{u} - \langle \nabla \mathbf{u}\rangle_{\tau B}|^2\,dx \\ &= c \fint_{\tau B} \varphi''(|\nabla \mathbf{u}| + |\langle \nabla \mathbf{u}\rangle_{\tau B}|)\,|\nabla \mathbf{w} - \langle \nabla \mathbf{w}\rangle_{\tau B}|^2\,dx. \end{aligned} \tag{6.11}$$

We claim that

$$|\nabla \mathbf{u}| + |\langle \nabla \mathbf{u}\rangle_{\tau B}| \sim |\mathbf{Q}|. \tag{6.12}$$

From (6.2) it is clear that $|\nabla \mathbf{u}| + |\langle \nabla \mathbf{u}\rangle_{\tau B}| \le c\,|\mathbf{Q}|$ on τB (using $\tau \le \frac{1}{2}$). To show the converse, we will show that $|\langle \nabla \mathbf{u}\rangle_{\tau B}| \ge c\,|\mathbf{Q}|$.

$$\begin{aligned} \varphi(|\mathbf{Q}|) &\le c\,\varphi''(|\mathbf{Q}|)|\mathbf{Q}|^2 \\ &\le c\varphi''(|\mathbf{Q}|)\,\big(|\langle \nabla \mathbf{u}\rangle_{\tau R} - \mathbf{Q}|^2 + |\langle \nabla \mathbf{u}\rangle_{\tau R}|^2\big) \\ &\le c\,\varphi''(|\mathbf{Q}|)\bigg(\fint_{\tau B} |\nabla \mathbf{u} - \mathbf{Q}|^2\,dx + |\langle \nabla \mathbf{u}\rangle_{\tau B}|^2\bigg) \\ &\le c\,\varphi''(|\mathbf{Q}|)\bigg(\tau^{-n} \fint_{B} |\nabla \mathbf{u} - \mathbf{Q}|^2\,dx + |\langle \nabla \mathbf{u}\rangle_{\tau B}|^2\bigg). \end{aligned}$$

Now, (6.3), our assumption $\Phi(\mathbf{u}, B) \le \varepsilon_0 \sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|)$, and (6.1) imply

$$\begin{aligned} \varphi(|\mathbf{Q}|) &\le c\,\big(\tau^{-n}\varepsilon_0 \sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u})| + \varphi''(|\mathbf{Q}|)|\langle \nabla \mathbf{u}\rangle_{\tau B}|^2\big) \\ &\le c\,\big(\tau^{-n}\varepsilon_0\,\varphi(|\mathbf{Q}|) + \varphi''(|\mathbf{Q}|)|\langle \nabla \mathbf{u}\rangle_{\tau B}|^2\big). \end{aligned}$$

Now, this is the first of two places where we need to choose $\varepsilon_0 > 0$ small (depending on τ). If $\tau^{-n}\varepsilon_0 < \frac{1}{2c}$, then we obtain (using $\varphi''(|\mathbf{Q}|)\,|\mathbf{Q}|^2 \sim \varphi(|\mathbf{Q}|)$) that

$$|\mathbf{Q}|^2 \leq c\,|\langle\nabla\mathbf{u}\rangle_{\tau R}|^2.$$

This was the missing step to prove (6.12). Now, combining (6.12) with (6.11) (using also $\varphi''(t)\,t^2 \sim \varphi(t)$ and $\Delta_2(\varphi) < \infty$) we get

$$\Phi(\mathbf{u}, \tau B) \leq c \fint_{\tau B} \varphi''(|\mathbf{Q}|)\,|\nabla\mathbf{w} - \langle\nabla\mathbf{w}\rangle_{\tau B}|^2\,dx. \tag{6.13}$$

We estimate

$$\begin{aligned}
&\fint_{\tau B} |\nabla\mathbf{w} - \langle\nabla\mathbf{w}\rangle_{\tau B}|^2\,dx \\
&\leq 2\fint_{\tau B} \big[|\nabla\mathbf{v} - \langle\nabla\mathbf{v}\rangle_{\tau B}|^2 + |\nabla\mathbf{v} - \nabla\mathbf{w}|^2\big]\,dx \\
&\leq c\bigg(\tau^2 \fint_{\frac{1}{4}B} |\nabla\mathbf{v} - \langle\nabla\mathbf{v}\rangle_{\frac{1}{4}B}|^2\,dx + \tau^{-n}\fint_{\frac{1}{4}B} |\nabla\mathbf{v} - \nabla\mathbf{w}|^2\,dx\bigg) \\
&\leq c\bigg(\tau^2 \fint_{\frac{1}{4}B} |\nabla\mathbf{w} - \langle\nabla\mathbf{w}\rangle_{\frac{1}{4}B}|^2\,dx + \tau^{-n}\fint_{\frac{1}{4}B} |\nabla\mathbf{v} - \nabla\mathbf{w}|^2\,dx\bigg).
\end{aligned}$$

Now, combining (6.13), the previous estimate, $\nabla\mathbf{w} = \nabla\mathbf{u} - \mathbf{Q}$, (6.10), and (6.3) we get

$$\begin{aligned}
\Phi(\mathbf{u}, \tau B) &\leq c\,\tau^2\varphi''(|\mathbf{Q}|)\fint_{\frac{1}{2}B} |\nabla\mathbf{u} - \mathbf{Q}|^2\,dx + c\tau^{-n}c\,\big(\varphi(|\mathbf{Q}|)\big)^{-\beta_2} \\
&\quad\times\bigg(\fint_{\frac{1}{2}B} |\mathbf{V}(\nabla\mathbf{u}) - \mathbf{V}(\mathbf{Q})|^2\,dx\bigg)^{1+\beta_2} \\
&\leq c\,\tau^2\Phi(\mathbf{u}, B) + c\,\tau^{-n}\big(\varphi(|\mathbf{Q}|)\big)^{-\beta_2}\big(\Phi(\mathbf{u}, B)\big)^{1+\beta_2}.
\end{aligned}$$

Using $\Phi(\mathbf{u}, B) \leq \varepsilon_0 \sup_{\frac{1}{2}B}\varphi(|\nabla\mathbf{u}|)$ and (6.1) we estimate

$$\Phi(\mathbf{u}, \tau B) \leq c\,\big(\tau^2 + \tau^{-n}\varepsilon_0^{\beta_2}\big)\Phi(\mathbf{u}, B).$$

Now, this is the second place where we need to choose $\varepsilon_0 > 0$ small (depending on τ). If we additionally assume $\varepsilon_0 \leq \tau^{\frac{2+n}{\beta_2}}$, then

$$\Phi(\mathbf{u}, \tau B) \leq c_2\,\tau^2\Phi(\mathbf{u}, B) \tag{6.14}$$

for some $c_2 > 0$. Note that c_2 does not depend on τ. □

Exactly as in [15, Proposition 3.2], we can remove the hypotheses in Lemma 6.2 by proving the following lemma.

Lemma 6.3. *Let φ satisfy Assumption 2.2 and let $\mathbf{u}$ be a local minimizer of the functional (1.1). Then there exists $c > 1$ such that for every $\tau \in (0, 1)$ there exists $\varepsilon_0 > 0$ and $\delta \in (0, 1)$ such that for every ball $B \Subset \Omega$ one of the following alternatives holds*:

(a) $\Phi(\mathbf{u}, \tau B) \le c\,\tau^2\,\Phi(\mathbf{u}, B)$,
(b) $\Phi(\mathbf{u}, B) > \varepsilon_0 \sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|)$ *and* $\sup_{\frac{1}{4}B} \varphi(|\nabla \mathbf{u}|) \le \delta \sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|)$.

Proof. Fix $\tau \in (0, 1)$. Without loss of generality we can assume $\tau < \frac{1}{4}$. Choose $\varepsilon_0 > 0$ as in Lemma 6.2. We will choose $\delta > 0$ later.

If the second alternative (b) is not true, then

$$\Phi(\mathbf{u}, B) \le \varepsilon_0 \sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|) \tag{6.15}$$

or

$$\Phi(\mathbf{u}, B) > \varepsilon_0 \sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|) \quad \text{and} \quad \sup_{\frac{1}{4}B} \varphi(|\nabla \mathbf{u}|) > \delta \sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|). \tag{6.16}$$

In the case of (6.15) we deduce from Lemma 6.2 that (a) holds. So in the following assume that (6.16) holds. The Harnack inequality (5.17) then yields

$$\begin{aligned}
\Phi\Big(\mathbf{u}, \frac{1}{4}B\Big) &\le c\Big(\sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|) - \sup_{\frac{1}{4}B} \varphi(|\nabla \mathbf{u}|)\Big) \\
&\le c\,(1-\delta) \sup_{\frac{1}{2}B} \varphi(|\nabla \mathbf{u}|) \le c\,\frac{1-\delta}{\varepsilon_0}\Phi(\mathbf{u}, B)
\end{aligned}$$

and therefore

$$\Phi(\mathbf{u}, \tau B) \le c\,\tau^{-n}\Phi\Big(\mathbf{u}, \frac{1}{4}B\Big) \le c\tau^{-n}\frac{1-\delta}{\varepsilon_0}\Phi(\mathbf{u}, B)$$

So choosing δ close to 1 in such a way that $c\tau^{-n}\frac{1-\delta}{\varepsilon_0} < \tau^2$, we conclude again that

$$\Phi(\mathbf{u}, \tau B) \le \tau^2 \Phi(\mathbf{u}, B)$$

We notice that δ depends on τ, but not on B. □

Now, based on Lemma 6.3 the standard iteration technique (see the proof of Theorem 3.1 in [15]) allows us to conclude with the following theorem.

Theorem 6.4. *Let φ satisfy Assumption 2.2 and let $\mathbf{u}$ be a local minimizer of the functional (1.1). Then there exists $\sigma > 0$ such that for all balls $B \Subset \Omega$ and all $\lambda \in (0, 1)$ holds*

$$\Phi(\mathbf{u}, \lambda B) \le c\lambda^{\sigma}\Phi(\mathbf{u}, B).$$

From Campanato's characterization of Hölder continuous functions, see chapter III of [14], we immediately conclude the local σ-Hölder continuity of $\mathbf{V}(\nabla \mathbf{u})$. So together with Lemma 2.10 we get the following result.

Theorem 6.5. *Let φ satisfy Assumption 2.2 and let $\mathbf{u}$ be a local minimizer of the functional (1.1). Then there exists $\sigma > 0$ such that $\mathbf{V}(\nabla \mathbf{u})$, $\nabla \mathbf{u}$, and $\mathbf{A}(\nabla \mathbf{u})$ are locally σ-Hölder continuous.*

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