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Calderón-Zygmund Estimates for Systems of φ-Growth*

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We prove higher integrability results for the gradients of weak solutions of certain elliptic systems with φ-growth, where φ is a suitable convex function.

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1. Introduction

Recently there has been a rather large interest in proving gradient integrability estimates-Calderón-Zygmund estimates- for elliptic and parabolic problems ([1], [2], [13]) especially under non-standard growth conditions. The aim of this paper is to provide further, more general results in this direction.

Let φ be a convex, C^1 function. Let us consider the nonhomogeneous elliptic system of φ-growth

$$\operatorname{div}\left(\varphi'(|Du|)\frac{Du}{|Du|}\right) = \operatorname{div}\left(\varphi'(|F|)\frac{F}{|F|}\right) \tag{1}$$

where $u \in W^{1,\varphi}(\mathbb{R}^n, \mathbb{R}^N)$ and $F \in L^{\varphi}(\mathbb{R}^n, \mathbb{R}^{nN})$. The standard examples for convex functions φ are

$$\varphi(t) = \int_0^t (\kappa + s^2)^{\frac{p-2}{2}}\, s\, ds \quad \text{and} \quad \varphi(t) = \int_0^t (\kappa + s)^{p-2}\, s\, ds, \tag{2}$$

where $\kappa \geq 0$.

In this paper we are interested in studying how the regularity of F is reflected upon the regularity of the solution of the system (1). More precisely if $u \in W^{1,\varphi}(\mathbb{R}^n, \mathbb{R}^N)$ is a weak solution of (1) i.e.

$$\int_{\mathbb{R}^n} \frac{\varphi'(|Du|)}{|Du|}\langle Du, D\eta\rangle dx = \int_{\mathbb{R}^n} \frac{\varphi'(|F|)}{|F|}\langle F, D\eta\rangle dx \tag{3}$$

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$\forall \eta \in W^{1,\varphi}_{loc}(\mathbb{R}^n, \mathbb{R}^N)$ compactly supported in $\mathbb{R}^n$ and $F \in L^{\psi}(\mathbb{R}^n, \mathbb{R}^{nN})$ with ψ appropriately greater than φ (see Section 3), whether there exists a constant $c > 0$ such that

$$\int_{\mathbb{R}^n} \psi(|Du|)dx \leq c \int_{\mathbb{R}^n} \psi(|F|)dx.$$

In case of linear growth $\varphi(t) = t^2$ the global results follow from the well known Calderón-Zygmund theory (see [16]).

For the non-linear case we can refer to the papers due to Iwaniec [17] and DiBenedetto-Manfredi [10] in which they proved higher integrability results for the gradients of weak solutions of elliptic systems like (1) when $\varphi(t) = t^p$, $p > 1$. Later on a large number of generalizations have been made. In particular it is worth to notice recent advances in this direction due to Mingione [24].

Here we will extend the global results contained in [10], obtained in the power case, to the context of a more general growth condition.

The basic idea to prove the results is to compare the gradient of a solution u of our problem with the gradient of the solution of an homogeneous problem in a ball with given boundary data u. For homogeneous systems of φ-growth the $C^{1,\alpha}$ excess decay estimate has been proved in [14]. Important tools are the Hardy-Littlewood maximal function and the sharp maximal function of Fefferman-Stein, for which useful estimates are known.

We present in some case a unified approach to the superquadratic and subquadratic p-growth, considering more general functions than the powers.

Furthermore in Section 3 we will present also some global estimates in the space BMO.

Finally, in the last Section, we will focus our attention on a local higher integrability result for gradients of solutions to a class of homogeneous elliptic systems of the type

$$\operatorname{div} A(x, Du) = 0$$

where $A : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is a Carathéodory function satisfying non standard growth-conditions and in particular A is close, in a C^0 sense, to be a constant matrix. Here Ω is a bounded and open subset of $\mathbb{R}^n$.

This last result can be viewed as an extension of certain results [6], [18], [26] obtained in the scalar case.

It is worth to point out that, at the same time, we are able to quantify the improved higher integrability. The crucial point is that the degree of integrability may be much greater than the natural exponent.

Let us observe that the results presented in this paper rely on some technical lemmas that have been proved in a paper of Diening and Ettwein [11], where they get fractional estimates for non-differentiable elliptic systems with φ-growth.

2. Notation and Preliminaries

To simplify the notation, the letter c will denote any positive constant, which may vary throughout the paper. If $w \in L^1_{\rm loc}$, for any ball $B \subset \mathbb{R}^n$ we set

$$w_B = \frac{1}{|B|}\int_B w(x)dx = \fint_B w(x)dx, \tag{4}$$

where $|B|$ is the n-dimensional Lebesgue measure of B. For $\lambda > 0$ we denote by λB the ball with the same center as B but λ-times the radius. The following definitions and results are standard in the context of N-functions. A real function $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is said to be an N-function if it satisfies the following conditions: $\varphi(0) = 0$ and there exists the derivative φ' of φ. This derivative is right continuous, non-decreasing and satisfies $\varphi'(0) = 0$, $\varphi'(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} \varphi'(t) = \infty$. Especially, φ is convex.

We say that φ satisfies the Δ_2–condition, if there exists $c_1 > 0$ such that for all $t \geq 0$ holds $\varphi(2t) \leq c_1\, \varphi(t)$. By $\Delta_2(\varphi)$ we denote the smallest constant c_1.

Since $\varphi(t) \leq \varphi(2t)$ the Δ_2 condition is equivalent to $\varphi(2t) \sim \varphi(t)$. For a family $\{\varphi_\lambda\}_\lambda$ of N-functions we define $\Delta_2(\{\varphi_\lambda\}_\lambda) := \sup_\lambda \Delta_2(\varphi_\lambda)$.

By L^φ and $W^{1,\varphi}$ we denote the classical Orlicz and Sobolev-Orlicz spaces, i.e. $f \in L^\varphi$ iff $\int \varphi(|f|)\, dx < \infty$ and $f \in W^{1,\varphi}$ iff $f, Df \in L^\varphi$. Let us observe that if $\Delta_2(\varphi) < \infty$ then the closure of C^∞_c in *strong*-L^φ coincides with the space L^φ.

By $(\varphi')^{-1} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ we denote the function

$$(\varphi')^{-1}(t) := \sup\left\{u \in \mathbb{R}^{\geq 0} \,:\, \varphi'(u) \leq t\right\}.$$

If φ' is strictly increasing then $(\varphi')^{-1}$ is the inverse function of φ'. Then $\varphi^* : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with

$$\varphi^*(t) := \int_0^t (\varphi')^{-1}(s)\, ds$$

is again an N-function and $(\varphi^*)'(t) = (\varphi')^{-1}(t)$ for $t > 0$. It is the complementary function of φ. Note that $\varphi^*(t) = \sup_{u\geq 0}(ut - \varphi(u))$ and $(\varphi^*)^* = \varphi$. For all $\delta > 0$ there exists c_δ (only depending on $\Delta_2(\{\varphi, \varphi^*\})$ such that for all $t, u \geq 0$ holds

$$t\, u \leq \delta\, \varphi(t) + c_\delta\, \varphi^*(u). \tag{5}$$

This inequality is called Young's inequality. For all $t \geq 0$

$$\begin{aligned} \frac{t}{2}\varphi'\left(\frac{t}{2}\right) &\leq \varphi(t) \leq t\, \varphi'(t), \\ \varphi\left(\frac{\varphi^*(t)}{t}\right) &\leq \varphi^*(t) \leq \varphi\left(\frac{2\,\varphi^*(t)}{t}\right). \end{aligned} \tag{6}$$

Therefore, uniformly in $t \geq 0$

$$\varphi(t) \sim \varphi'(t)\, t, \qquad \varphi^*\left(\varphi'(t)\right) \sim \varphi(t), \tag{7}$$

where the constants only depend on $\Delta_2(\{\varphi, \varphi^*\})$. Throughout the paper we will assume φ satisfies the following assumptions.

Assumption 2.1. Let φ be an N-function such that φ is C^1 on $[0,\infty)$ and C^2 on $(0,\infty)$. Further assume that

$$\varphi'(t) \sim t\,\varphi''(t) \tag{8}$$

uniformly in $t > 0$.

We remark that under these assumptions $\Delta_2(\{\varphi,\varphi^*\}) < \infty$ will be automatically satisfied, where $\Delta_2(\{\varphi,\varphi^*\})$ depends only on the constant in (8).

Assumption 2.2. Let φ be as in Assumption 2.1 such that there exist $\beta \in (0,1]$ and $c > 0$ such that

$$|\varphi''(s+t) - \varphi''(t)| \le c\,\varphi''(t)\left(\frac{|s|}{t}\right)^{\beta} \tag{9}$$

for all $t > 0$ and $s \in \mathbb{R}$ with $|s| < \frac{1}{2}t$.

We notice that assumption (9) is satisfied for example in the following three cases.

$$\begin{aligned}
&\varphi(t) = t^p,\\
&\varphi(t) = t^p \log^{\beta}(e+t), \qquad\qquad \beta > 0,\\
&\varphi(t) = t^p \log\log(e+t),
\end{aligned}$$

with $1 < p < \infty$.

For given φ we define the associated N-function χ by

$$\chi'(t) := \sqrt{\varphi'(t)\,t}\,. \tag{10}$$

Note that

$$\chi''(t) = \frac{1}{2}\left(\frac{\varphi''(t)}{\varphi'(t)} + 1\right)\sqrt{\frac{\varphi'(t)}{t}} = \frac{1}{2}\left(\frac{\varphi''(t)}{\varphi'(t)} + 1\right)\frac{\chi'(t)}{t}. \tag{11}$$

It is shown in [11, Lemma 25] that if φ satisfies Assumption 2.1, then also φ^*, χ, and χ^* satisfy this assumption and $\chi''(t) \sim \sqrt{\varphi''(t)}$.

Define $A, V : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ in the following way:

$$A(Q) = \varphi'(|Q|)\frac{Q}{|Q|}, \tag{12a}$$

$$V(Q) = \chi'(|Q|)\frac{Q}{|Q|}. \tag{12b}$$

For $\lambda \ge 0$ we define the *shifted N-function* φ_λ by $\varphi_\lambda(t) = \int_0^t \varphi'_\lambda(s)\,ds$ with

$$\varphi'_\lambda(t) := \frac{\varphi'(\lambda+t)}{\lambda+t}\,t \tag{13}$$

for $t > 0$. The shifted N-functions have been introduced in [11]. See [12] for a detailed study of the shifted N-functions.

The connection between A, V, and $\{\varphi_\lambda\}_{\lambda\ge 0}$ is best reflected in the following lemma (see [11]).

Lemma 2.3. *Let φ satisfy Assumption 2.1 and let A and V be defined by (12). Then*

$$\begin{aligned}(A(P)-A(Q))\cdot(P-Q) &\sim |V(P)-V(Q)|^2 &\text{(14a)}\\ &\sim \varphi_{|P|}(|P-Q|), &\text{(14b)}\\ &\sim |P-Q|^2\,\varphi''(|P|+|Q|)\,, &\text{(14c)}\end{aligned}$$

and

$$\begin{aligned}|A(P)-A(Q)| &\sim \varphi'_{|P|}(|P-Q|) &\text{(14d)}\\ &\sim \varphi''(|P|+|Q|)\,|P-Q| &\text{(14e)}\end{aligned}$$

uniformly in $P,Q\in\mathbb{R}^{nN}$. Moreover,

$$\begin{aligned}A(Q)\cdot Q \sim |V(Q)|^2 &\sim \varphi(|Q|), &\text{(14f)}\\ |A(Q)| &\sim \varphi'(|Q|) &\text{(14g)}\end{aligned}$$

uniformly in $Q\in\mathbb{R}^{nN}$.

Note that if $\varphi''(0)$ does not exists, the expression in (14c) and (14e) are continuously extended by zero for $|P|=|Q|=0$.

An immediate consequence of (13) is the following estimate:

$$\varphi(|P-Q|)\le c\varphi_{|P|}(|P-Q|)+c\varphi(|P|)\quad \forall P,\ Q\in\mathbb{R}^{nN}. \tag{15}$$

In fact $\forall\delta\in(0,1)$ we get

$$\begin{aligned}&\varphi(|P-Q|)\le c\varphi'(|P-Q|)|P-Q| = c\frac{\varphi'(|P-Q|)|P-Q|}{|P|+|P-Q|}\cdot(|P|+|P-Q|)\\ &= c\frac{\varphi'(|P-Q|)}{|P|+|P-Q|}\cdot|P-Q|^2 + c\frac{\varphi'(|P-Q|)|P-Q|}{|P|+|P-Q|}\cdot(|P|)\\ &\le c\frac{\varphi'(|P|+|P-Q|)}{|P|+|P-Q|}\cdot|P-Q|^2 + c\varphi'(|P-Q|)\cdot(|P|)\\ &\le c\varphi_{|P|}(|P-Q|)+c_\delta\varphi(|P|)+\delta\varphi(|P-Q|),\end{aligned}$$

where we have used Young's inequality and (13). Finally absorbing on the left hand side the last term in the right hand side we obtain (15).

Let us observe that the functions φ_λ with $\lambda\ge 0$ share the same properties of φ. In particular, the following result holds (see [11]):

Lemma 2.4. *Let φ satisfy Assumption 2.1. Then for all $\lambda\ge0$ the function φ_λ satisfies Assumption 2.1 and $\Delta_2(\{\varphi_\lambda\}_{\lambda\ge0},\{(\varphi_\lambda)^*\}_{\lambda\ge0})<\infty$. Moreover*

$$\varphi''_\lambda(t)\sim\varphi''(\lambda+t)\sim\frac{\varphi'(\lambda+t)}{\lambda+t}\sim\frac{\varphi'_\lambda(t)}{t},$$

uniformly in $\lambda, t \geq 0$ *with* $\lambda + t > 0$. *In particular,* φ_λ *satisfies Assumption* 2.1 *with constants independent of* $\lambda \geq 0$. *If* $\lambda > 0$, *then* φ_λ *is* C^2 *on* $[0, \infty)$.

If φ *satisfies Assumption* 2.2, *then* φ_λ *satisfies Assumption* 2.2 *and the constant in* (9) *does not depend on* λ.

Since $(\varphi^*)'$ is the inverse of φ', it follows that

$$A^{-1}(Q) = (\varphi^*)'(|Q|)\frac{Q}{|Q|} \tag{16}$$

for all $Q \in \mathbb{R}^{N\times n}$. In particular, A is invertible. The same holds for A_λ and V_λ.

Now, let φ, φ_1 be N-functions. We shall write

$$\varphi_1 \succ \varphi$$

if $\bar{\varphi} = \varphi_1 \circ \varphi^{-1}$ is an N-function too.

Remark 2.5. Next let us recall that if a function φ satisfies Assumption 2.1, then there exist $q \geq p > 1$ such that $\varphi(t)/t^p$ is increasing and $\varphi(t)/t^q$ is decreasing. We refer to the exponents p and q as lower and upper index of φ respectively (see [7], [9], [21] for more details).

If $f \in L^1(\mathbb{R}^n)$ let $\mathcal{M}f(\cdot)$ the maximal function of Hardy-Littlewood defined as follows

$$\mathcal{M}f(x) = \sup_{\rho>0} ⨍_{B_\rho(x)} |f| dx, \;\; x \in \mathbb{R}^n$$

and the sharp maximal function of Fefferman-Stein

$$f^\sharp(x) = \sup_{\rho>0} ⨍_{B_\rho(x)} |f - (f)_{x,\rho}| dx, \;\; x \in \mathbb{R}^n.$$

The following version of the Fefferman and Stein inequality holds:

Lemma 2.6. *Let* $\Phi : [0, \infty) \to [0, \infty)$ *be an increasing and continuous function with* $\Phi(0) = 0$. *Let* $\epsilon > 0$ *and* $f : \mathbb{R}^n \to \mathbb{R}^{nN}$ *be a compactly supported and essentially bounded Borel map. Then*

$$\begin{aligned} &\int_{\mathbb{R}^n} \Phi(|\mathcal{M}f|) dx \\ &\leq \frac{5^N}{\epsilon} \int_{\mathbb{R}^n} \Phi\left(\frac{|f^\sharp|}{\epsilon}\right) dx + 2 \cdot 5^{3n}\epsilon \int_{\mathbb{R}^n} \Phi\left(5^n \cdot 2^{n+1} \cdot |\mathcal{M}f|\right) dx. \end{aligned} \tag{17}$$

(see [20] and [26] for a local version).

Now let us recall that if f is such that $f^\sharp$ is bounded, we say that f is a function of bounded mean oscillation and we denote by BMO the space formed by these functions. For f in BMO we write

$$\|f\|_{BMO} = \|f^\sharp\|_\infty.$$

Note that the null elements in the BMO norm are the constants, so that a function in BMO is defined only up to an additive constant (see [25] for more details).

Let us consider $f \in L^{\varphi}_{loc}(\mathbb{R}^n)$, with φ verifying Assumption 2.1. Let $M^{\sharp}_{\varphi}(f,x_0)$ denote the maximal function defined by

$$M^{\sharp}_{\varphi}(f,x_0) = \sup_{\rho>0} \varphi^{-1}\left(\fint_{B_\rho(x_0)} \varphi(|f-(f)_{x_o,\rho}|)dx \right), \quad x_0 \in \mathbb{R}^n.$$

Let us observe that the BMO-norm of f satisfies

$$\|f\|_{BMO} \le \|M^{\sharp}_{\varphi}(f,\cdot)\|_\infty \le c\|f\|_{BMO}. \tag{18}$$

In fact let us recall the well known John-Nirenberg inequality for a function $f \in BMO$:

$$|\{x \in B : |f(x)-f_B| > \alpha\}| \le ce^{\frac{-c\alpha}{\|f\|_{BMO}}} \cdot |B|$$

for every $\alpha > 0$ and every ball B, with c positive constant. We get the following

$$\begin{aligned}
&\int_B \varphi(|f-(f)_B|)dx = \int_0^\infty \varphi'(t)|\{x\in B : |f(x)-f_B|>t\}|dt \\
&\le c\int_0^\infty \varphi'(t)e^{\frac{-c_2 t}{\|f\|_{BMO}}}dt\cdot|B| = c|B|\int_0^\infty \varphi'\left(\frac{s\|f\|_{BMO}}{c}\right)e^{-s}\cdot\left(\frac{\|f\|_{BMO}}{c}\right)ds \\
&\le c|B|\left(\varphi\left(\frac{\|f\|_{BMO}}{c}\right)\int_0^1 e^{-s}ds + \varphi\left(\frac{\|f\|_{BMO}}{c}\right)\int_1^\infty s^{q-1}\cdot e^{-s}ds\right) \\
&\le c|B|\varphi(\|f\|_{BMO})
\end{aligned}$$

where q is the upper index of φ as in the Remark 2.5.

3. The global L^ψ-estimate

We will prove a global estimate for the gradient of weak solutions to the nonlinear degenerate system (1).

We will denote by $D^\varphi(\mathbb{R}^n,\mathbb{R}^N)$ the space of distributions whose gradient is in $L^\varphi(\mathbb{R}^n,\mathbb{R}^{nN})$.

Theorem 3.1. *Let $u \in D^\varphi(\mathbb{R}^n,\mathbb{R}^N)$ be a weak solution of* (1) *in $\mathbb{R}^n$, where φ satisfies Assumption* 2.2. *If $F \in L^\psi(\mathbb{R}^n,\mathbb{R}^{nN})$, for an N-function $\psi \succ \varphi$ such that*

$$\Delta_2(\psi) < \infty \quad \textit{and} \quad \Delta_2\left((\psi\circ\varphi^{-1})^*\right) < \infty, \tag{19}$$

then $|Du| \in L^\psi(\mathbb{R}^n)$ and there exists a constant c depending only on n, φ, ψ such that

$$\|Du\|_{L^\psi} \le c\|F\|_{L^\psi}. \tag{20}$$

If $x_0 \in \mathbb{R}^n$ let $B_R(x_0) \subset \mathbb{R}^n$, $R > 0$ and let $v \in W^{1,\varphi}(B_R(x_0),\mathbb{R}^N)$ be the solution of the problem

$$\begin{cases} \operatorname{div} \dfrac{\varphi'(|Dv|)Dv}{|Dv|} = 0 & \text{in } B_R(x_0) \\ v = u & \text{on } \partial B_R(x_0). \end{cases} \tag{21}$$

Let us observe that the existence of a unique solution of the problem (21) is due to the fact that since the map $w \to -\operatorname{div}(A(D(w+u)))$, with A defined in (12a) and u solution of (1), is strictly monotone and hemicontinuous from $W_0^{1,\varphi}(B_R(x_0),\mathbb{R}^N)$ to $W^{-1,\varphi}(B_R(x_0),\mathbb{R}^N)$, there exists a unique solution $w \in W_0^{1,\varphi}(B_R(x_0),\mathbb{R}^N)$ such that

$$\int_{B_R(x_0)} \langle A(D(w+u)), D\xi\rangle = 0$$

for all $\xi \in W_0^{1,\varphi}(B_R(x_0),\mathbb{R}^N)$. Therefore if we define $v := w+u$, then $v \in W^{1,\varphi}(B_R(x_0),\mathbb{R}^N)$, $v = u$ on $\partial B_R(x_0)$ and

$$\int_{B_R(x_0)} \langle A(D(v)), D\xi\rangle = 0$$

for all $\xi \in W_0^{1,\varphi}(B_R(x_0),\mathbb{R}^N)$.

The following facts concerning v can be found in [14]. First of all

$$\int_{B_R(x_0)} \varphi(|Dv|)dx \le \int_{B_R(x_0)} \varphi(|Du|)dx. \tag{22}$$

There exists a constant $c = c(n,\varphi)$ such that $\forall x_0 \in \mathbb{R}^n$, $\forall 0 < \rho \le \frac{R}{2}$

$$\sup_{B_\rho(x_0)} \varphi(|Dv|) \le c \int_{B_R(x_0)} \varphi(|Dv|)dx. \tag{23}$$

There exists constants $\alpha \in (0,1)$, $c > 1$ depending only on n, φ such that such that $\forall x_0 \in \mathbb{R}^n$, $\forall 0 < \rho \le R$

$$\begin{aligned} &\fint_{B_\rho(x_0)} |V(Dv) - (V(Dv))_{x_0,\rho}|^2\, dx \\ &\le c\left(\frac{\rho}{R}\right)^\alpha \fint_{B_R(x_0)} |V(Dv) - (V(Dv))_{x_0,R}|^2\, dx \end{aligned} \tag{24}$$

(see Theorem 1.1 in [14]).

Lemma 3.2. *Fix an arbitrary $\epsilon \in (0,1)$. There exist constants $\gamma_\epsilon = \gamma_\epsilon(n,\varphi,\epsilon)$ such that $\forall x_0 \in \mathbb{R}^n$, $\forall 0 < \rho \le hR$*

$$\begin{aligned} &\fint_{B_\rho(x_0)} |V(Du) - V(Dv)|^2 dx \\ &\le \gamma_\epsilon \left(\frac{R}{\rho}\right)^n \fint_{B_R(x_0)} \varphi(|F|)dx + \epsilon\left(\frac{R}{\rho}\right)^n \fint_{B_R(x_0)} \varphi(|Du|)dx. \end{aligned} \tag{25}$$

Proof. After a translation we may assume $x_0 = 0$. Subtract (21) from (1) and let us take the test function $w = u - v$ in the weak formulation, to obtain

$$\int_{B_R} \left\langle \frac{\varphi'(|Du|)}{|Du|} Du - \frac{\varphi'(|Dv|)}{|Dv|} Dv, Dw \right\rangle = \int_{B_R} \left\langle \frac{\varphi'(|F|)}{|F|} F, Dw \right\rangle. \tag{26}$$

Since (14a) we can estimate the left hand side from below as

$$\int_{B_R} |V(Du) - V(Dv)|^2 dx \geq \int_{B_\rho} |V(Du) - V(Dv)|^2 dx.$$

On the other hand we can estimate the right hand side of (26) using Young's inequality and (7)

$$\begin{aligned} & c\int_{B_R} \varphi(|F|)dx + \epsilon \int_{B_R} \varphi(|Du - Dv|)dx \\ & \leq c\int_{B_R} \varphi(|F|)dx + \epsilon \int_{B_R} \varphi(|Du|)dx \end{aligned}$$

where we have used (22). Therefore considering the integral mean we get

$$\begin{aligned} & \fint_{B_\rho(x_0)} |V(Du) - V(Dv)|^2 dx \\ & \leq \gamma_\epsilon \left(\frac{R}{\rho}\right)^n \fint_{B_R(x_0)} \varphi(|F|)dx + \epsilon \left(\frac{R}{\rho}\right)^n \fint_{B_R(x_0)} \varphi(|Du|)dx. \end{aligned}$$

□

Proposition 3.3. *Fix an arbitrary $\delta \in (0,1)$. There exist constants $\gamma_\delta = \gamma_\delta(n, \varphi, \delta)$ and $h = h(n, \varphi, \delta) \in (0,1)$ such that $\forall x_0 \in \mathbb{R}^n$, $\forall 0 < \rho \leq hR$*

$$\begin{aligned} & \fint_{B_\rho(x_0)} \left| |V(Du)|^2 - (|V(Du)|^2)_{x_0,\rho} \right| dx \\ & \leq \gamma_\delta \left(\frac{R}{\rho}\right)^n \fint_{B_R(x_0)} \varphi(|F|)dx + \delta \left(\frac{R}{\rho}\right)^n \fint_{B_R(x_0)} \varphi(|Du|)dx. \end{aligned} \tag{27}$$

Proof. Assume as before that $x_0 = 0$. Then for all $\rho \in (0, hR)$ and for every $\eta \in (0,1)$ we have

$$\begin{aligned} & \fint_{B_\rho(x_0)} \left| |V(Du)|^2 - (|V(Du)|^2)_{x_0,\rho} \right| dx \\ & \leq c \fint_{B_\rho(x_0)} \left| |V(Du)|^2 - |(V(Dv))_{x_0,\rho}|^2 \right| dx \\ & \leq c \fint_{B_\rho(x_0)} \left| |V(Du)|^2 - |V(Dv)|^2 \right| dx + c \fint_{B_\rho(x_0)} \left| |V(Dv)|^2 - |(V(Dv))_{x_0,\rho}|^2 \right| dx \\ & \leq c \Bigg(\fint_{B_\rho(x_0)} |V(Du) - V(Dv)|(|V(Du)| + |V(Dv)|)dx \\ & \qquad + \fint_{B_\rho(x_0)} |V(Dv) - (V(Dv))_{x_0,\rho}|(|V(Dv)| + |(V(Dv))_{x_0,\rho}|)dx \Bigg) \\ & \leq c_\delta \fint_{B_\rho(x_0)} |V(Du) - V(Dv)|^2 dx + c_\delta \fint_{B_\rho(x_0)} |V(Dv) - V(Dv)_{x_0,\rho}|^2 dx \\ & \qquad + \delta \left(\frac{R}{\rho}\right)^n \fint_{B_R(x_0)} \varphi(|Du|)dx \\ & = I + II + \delta \left(\frac{R}{\rho}\right)^n \fint_{B_R(x_0)} \varphi(|Du|)dx, \end{aligned}$$

where we have used Young's inequality and (22). By Lemma 3.2 choosing ϵ small enough we estimate

$$I \leq \gamma_\delta \left(\frac{R}{\rho}\right)^n \fint_{B_R(x_0)} \varphi(|F|)dx + \delta \left(\frac{R}{\rho}\right)^n \fint_{B_R(x_0)} \varphi(|Du|)dx.$$

On the other hand we can estimate II with (24) as follows

$$II \leq c \left(\frac{\rho}{R}\right)^\alpha \fint_{B_R(x_0)} \left|V(Dv) - (V(Dv))_{x_0,R}\right|^2 dx \tag{28}$$

Now we can conclude that for every $\delta \in (0,1)$ there exist constants $\gamma_\delta = \gamma_\delta(n, \varphi, \delta)$ and $c = c(n, \varphi)$ such that

$$\begin{aligned} &\fint_{B_\rho(x_0)} \left||V(Du)|^2 - (|V(Du)|^2)_{x_0,\rho}\right| dx \\ &\leq \gamma_\delta \left(\frac{R}{\rho}\right)^n \fint_{B_R(x_0)} \varphi(|F|)dx + \left\{\delta \left(\frac{R}{\rho}\right)^n + c \left(\frac{\rho}{R}\right)^\alpha\right\} \fint_{B_R(x_0)} \varphi(|Du|)dx, \end{aligned}$$

where we have used again (22). □

Proof of Theorem 3.1. From Proposition 3.3, taking $\rho = \frac{R}{2}$, for every $\delta \in (0,1)$ there exists a constant $\gamma_\delta = \gamma_\delta(n, \varphi, \delta)$ such that

$$(|V(Du)|^2)^\sharp(x_0) \leq \gamma_\delta \mathcal{M}[\varphi(|F|)](x_0) + \delta \mathcal{M}[\varphi(|Du|)](x_0), \quad \text{a.e. } x_0 \in \mathbb{R}^n. \tag{29}$$

Step 1. Assume momentarily that

$$|Du| \in L^\psi(\mathbb{R}^n), \qquad F \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{nN}). \tag{30}$$

If we define $\Phi := \psi \circ \varphi^{-1}$ we get

$$\int_{\mathbb{R}^n} \psi(|Du|)dx = \int_{\mathbb{R}^n} \Phi(\varphi(|Du|))dx \leq c \int_{\mathbb{R}^n} \Phi(\mathcal{M}(\varphi(|Du|))dx$$

where we have used the definition of the Hardy-Littlewood maximal function. Since (17) and (19) we get that there exists a constant $c = c(n, \varphi, \psi)$ such that

$$\begin{aligned} &\int_{\mathbb{R}^n} \Phi(\mathcal{M}(|V(Du)|^2))dx \\ &\leq c\gamma_\epsilon \int_{\mathbb{R}^n} \Phi((|V(Du)|^2)^\sharp)dx + c\epsilon \int_{\mathbb{R}^n} \Phi(\mathcal{M}(|V(Du)|^2))dx. \end{aligned}$$

Now using (29) and choosing ϵ small enough we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} \Phi(\mathcal{M}(|V(Du)|^2))dx \\ &\leq c\gamma_\delta \int_{\mathbb{R}^n} \Phi(\mathcal{M}(\varphi(|F|)))dx + c\delta \int_{\mathbb{R}^n} \Phi(\mathcal{M}(|V(Du)|^2))dx. \end{aligned}$$

Thanks to the assumption (19) we can apply the Hardy-Littlewood maximal theorem in the Orlicz spaces (see for instance [19]) in order to obtain

$$\int_{\mathbb{R}^n} \Phi(\mathcal{M}(|V(Du)|^2))dx \leq c \int_{\mathbb{R}^n} \Phi(\varphi(|F|))dx$$

which implies together with (30) that

$$\int_{\mathbb{R}^n} \psi(|Du|)dx \leq c \int_{\mathbb{R}^n} \Phi(\varphi(|F|))dx = C \int_{\mathbb{R}^n} \psi(|F|)dx,$$

where $c = c\,(n, \varphi, \psi)$.

Step 2. Let us remove the assumption (30) using an approximation argument.

Let $F_h \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{nN})$ be a sequence converging to F in $L^\psi(\mathbb{R}^n, \mathbb{R}^{nN})$ and in $L^\varphi(\mathbb{R}^n, \mathbb{R}^{nN})$, and let $u_h \in W^{1,\varphi}(\mathbb{R}^n, \mathbb{R}^N)$ be the unique solution of

$$\operatorname{div}\left(\varphi'(|Du_h|)\frac{Du_h}{|Du_h|}\right) = \operatorname{div}\left(\varphi'(|F_h|)\frac{F_h}{|F_h|}\right). \tag{31}$$

Since F_h is compactly supported in $\mathbb{R}^n$, we have

$$\operatorname{div}\left(\varphi'(|Du_h|)\frac{Du_h}{|Du_h|}\right) = 0 \ \text{ in } \{|x| > k\}$$

for sufficiently large k. Next we prove that $|Du_h| \in L^\psi(\mathbb{R}^n)$ like in [10]. By inequality (23), for all $|x| > 2k$

$$\varphi(|Du_h|)(x) \leq c\left(\fint_{B_{|x|-k}(x)} \varphi(|Du_h|)(y)dy\right) \leq \frac{c}{(|x|-k)^n}\int_{\mathbb{R}^n} \varphi(|Du_h|)(y)dy.$$

This and inequality (20) imply that

$$\begin{aligned}\int_{\{|x|>2k\}} \psi(|Du_h|)(y)dy &\leq \int_{\{|x|>2k\}} \Phi\left(\frac{c}{(|x|-k)^n}\right)dx \\ &\leq c(n,\varphi,\psi)\int_{2k}^{\infty}\left(\frac{1}{(r-k)^{ns}}\right)r^{n-1}dr < \infty\end{aligned}$$

where $s > 1$ is the lower index of Φ. In fact the assumption (19) implies that the function Φ has a lower index $s > 1$ (see [21]). Thus since u_h verifies the system in (31) and $F_h \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^{nN})$ we get that $|Du_h|$ is locally bounded and it is in $L^\psi(\mathbb{R}^n)$. From *Step 1* we gain

$$\int_{\mathbb{R}^n} \psi(|Du_h|)dx \leq c\int_{\mathbb{R}^n} \psi(|F_h|)dx$$

where $c = c(n, \varphi, \psi)$. It remains to show that $Du_h \to Du$ a.e. in $\mathbb{R}^n$, in order to apply Fatou's Lemma. To this aim let us subtract (31) from (1) and take $u - u_h$ as test function to obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\left\langle \frac{\varphi'(|Du|)}{|Du|}Du - \frac{\varphi'(|Du_h|)}{|Du_h|}Du_h, Du - Du_h\right\rangle \\ &= \int_{B_R}\left\langle \frac{\varphi'(|F|)}{|F|}F - \frac{\varphi'(|F_h|)}{|F_h|}F_h, Du - Du_h\right\rangle.\end{aligned} \tag{32}$$

Since (14a) we can estimate the left hand side from below as

$$\int_{\mathbb{R}^n} |V(Du) - V(Du_h)|^2 dx.$$

On the other hand we can estimate the right hand side of (32) with (14d) to obtain

$$\int_{\mathbb{R}^n} \varphi'_{|F|}\left(|F - F_h|\right)|Du_h - Du|dx.$$

Now we can use Young's inequality and the equivalence (7) with $\varphi^*_{|F|}$, $\varphi_{|F|}$ and $\varphi'_{|F|}$ instead of φ^*, φ, φ' (see Lemma 32 and Lemma 34 in [11]). We get that for all $\epsilon > 0$ there exists C_ϵ such that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varphi'_{|F|}\left(|F - F_h|\right)|Du_h - Du|dx \\
&\leq C_\epsilon \int_{\mathbb{R}^n} \varphi^*_{|F|}\left(\varphi'_{|F|}\left(|F - F_h|\right)\right)dx + \epsilon \int_{\mathbb{R}^n} \varphi_{|F|}(|Du_h - Du|)dx \\
&\leq C_\epsilon \int_{\mathbb{R}^n} \varphi_{|F|}\left(|F - F_h|\right)dx + \epsilon \int_{\mathbb{R}^n} \varphi_{|F|}(|Du_h - Du|)dx \\
&\leq C_\epsilon \int_{\mathbb{R}^n} \varphi_{|F|}\left(|F - F_h|\right)dx + \epsilon \int_{\mathbb{R}^n} \varphi'\left(|F| + |Du_h - Du|\right)\frac{|Du_h - Du|^2}{|F| + |Du_h - Du|}dx \\
&\leq C_\epsilon \int_{\mathbb{R}^n} \varphi_{|F|}\left(|F - F_h|\right) + \epsilon \int_{\{|Du_h - Du| \leq |F|\}} \varphi'(2|F|)|Du_h - Du| \\
&\qquad + \epsilon \int_{\{|Du_h - Du| > |F|\}} \varphi'(2|Du_h - Du|)|Du_h - Du| \\
&\leq C_\epsilon \int_{\mathbb{R}^n} \varphi_{|F|}\left(|F - F_h|\right)dx + \epsilon \int_{\mathbb{R}^n} \varphi'(2|F|)|Du_h - Du|dx \\
&\qquad + \epsilon \int_{\mathbb{R}^n} \varphi(2|Du_h - Du|)dx.
\end{aligned}$$

Using again Young's inequality applied to φ and φ^* and using (7) we arrive to

$$\begin{aligned}
&C_\epsilon \int_{\mathbb{R}^n} \varphi_{|F|}\left(|F - F_h|\right)dx + c\epsilon \int_{\mathbb{R}^n} \varphi(2|F|)dx + c\epsilon \int_{\mathbb{R}^n} \varphi(2|Du_h - Du|)dx \\
&\leq C_\epsilon \int_{\mathbb{R}^n} \varphi_{|F|}\left(|F - F_h|\right)dx + c\epsilon \int_{\mathbb{R}^n} \varphi(|F|)dx + c\epsilon \int_{\mathbb{R}^n} \varphi(|Du_h - Du|)dx \\
&\leq C_\epsilon \int_{\mathbb{R}^n} \varphi_{|F|}\left(|F - F_h|\right)dx + c\epsilon \int_{\mathbb{R}^n} \varphi(|F|)dx \\
&\qquad + c\epsilon \int_{\mathbb{R}^n} |V(Du_h) - V(Du)|^2 dx + c\epsilon \int_{\mathbb{R}^n} \varphi(|Du|)dx
\end{aligned}$$

where we have used (15) and (14b).

Therefore for ϵ small enough we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} |V(Du) - V(Du_h)|^2 dx \\
&\leq \frac{C_\epsilon}{1 - c\epsilon} \int_{\mathbb{R}^n} \varphi_{|F|}\left(|F - F_h|\right)dx + \frac{c\epsilon}{(1 - c\epsilon)} \int_{\mathbb{R}^n} \varphi(|F|) + \varphi(|Du|)dx.
\end{aligned}$$

Next letting h to ∞ and then ϵ to 0 we obtain

$$\int_{\mathbb{R}^n} |V(Du) - V(Du_h)|^2 dx \to 0.$$

In particular, up to a subsequence, we get $V(Du_h) \to V(Du)$ a.e. in $\mathbb{R}^n$. Now let us observe that V^{-1} is continuous (see Lemma 2.9 in [14]). As a consequence we have that $Du_h \to Du$ a.e. in $\mathbb{R}^n$ and then the Theorem will follow by Fatou's Lemma.

4. The global BMO-estimate

Theorem 4.1. *Let $u \in D^\varphi(\mathbb{R}^n, \mathbb{R}^N)$ be a weak solution of* (1) *in $\mathbb{R}^n$, where φ satisfies Assumption* 2.2 *and φ'' is an increasing function. If $\frac{\varphi'(|F|)F}{|F|} \in BMO(\mathbb{R}^n, \mathbb{R}^{nN})$, then $V(Du) \in BMO(\mathbb{R}^n, \mathbb{R}^{nN})$, and there exists a constant c depending only on n, φ, such that*

$$\|V(Du)\|^2_{BMO} \le c\varphi^* \left(\left\| \frac{\varphi'(|F|)F}{|F|} \right\|_{BMO} \right). \tag{33}$$

In order to prove this result will be useful to recall that, under the assumption φ'' increasing function, the following inequality holds:

$$\left\langle \frac{\varphi'(|\eta|)\eta}{|\eta|} - \frac{\varphi'(|\xi|)\xi}{|\xi|}, \eta - \xi \right\rangle \ge c\,\varphi''(|\eta - \xi|)|\eta - \xi|^2 \tag{34}$$

for all $\xi, \eta \in \mathbb{R}^n$ (see [26] for the proof).

Proof. As in Lemma 3.2 we get

$$\begin{aligned} &\int_{B_R} |V(Du) - V(Dv)|^2 dx \\ \le &\int_{B_R} \left\langle \frac{\varphi'(|Du|)}{|Du|} Du - \frac{\varphi'(|Dv|)}{|Dv|} Dv, Du - Dv \right\rangle \\ = &\int_{B_R} \left\langle \frac{\varphi'(|F|)}{|F|} F - \left(\frac{\varphi'(|F|)}{|F|} F \right)_{x_0,R}, Du - Dv \right\rangle dx \end{aligned}$$

where v is a weak solution of the system (21). Now we apply the Young's inequality to the right hand side in order to get $\forall \eta > 0$

$$C_\eta \int_{B_R} \varphi^* \left(\left| \frac{\varphi'(|F|)}{|F|} F - \left(\frac{\varphi'(|F|)}{|F|} F \right)_{x_0,R} \right| \right) dx + \eta \int_{B_R} \varphi(|Du - Dv|) dx.$$

Since φ'' is an increasing function, we can apply inequality (34) and (14c) to obtain

$$\int_{B_R} \varphi(|Du - Dv|) dx \le \int_{B_R} |V(Du) - V(Dv)|^2 dx.$$

Therefore since (18) we get

$$\int_{B_R} |V(Du) - V(Dv)|^2 dx \le c|B_R| \varphi^* \left(\left\| \frac{\varphi'(|F|)F}{|F|} \right\|_{BMO} \right).$$

Now $\forall\ 0<\rho<R$, using again (24) we obtain

$$\begin{aligned}
&\fint_{B_\rho}|V(Du)-(V(Du))_\rho|^2dx \le \fint_{B_\rho}|V(Du)-(V(Dv))_\rho|^2dx\\
&\le \fint_{B_\rho}|V(Du)-V(Dv)|^2dx+\fint_{B_\rho}|V(Dv)-(V(Dv))_\rho|^2dx\\
&\le c\left(\frac{R}{\rho}\right)^n\fint_{B_R}|V(Du)-V(Dv)|^2dx+c\left(\frac{\rho}{R}\right)^\alpha\fint_{B_R}|V(Dv)-(V(Dv))_R|^2dx\\
&\le c\left(\frac{R}{\rho}\right)^n\varphi^*\left(\left\|\frac{\varphi'(|F|)F}{|F|}\right\|_{BMO}\right)+c\left(\frac{\rho}{R}\right)^\alpha\fint_{B_R}|V(Du)-V(Dv)|^2dx\\
&\qquad +c\left(\frac{\rho}{R}\right)^\alpha\fint_{B_R}|V(Du)-(V(Du))_R|^2dx\\
&\le c\left(\frac{R}{\rho}\right)^n\varphi^*\left(\left\|\frac{\varphi'(|F|)F}{|F|}\right\|_{BMO}\right)+c\left(\frac{\rho}{R}\right)^\alpha\fint_{B_R}|V(Du)-(V(Du))_R|^2dx.
\end{aligned}$$

Next choosing $\rho=hR$, $h\in(0,1)$ we arrive to

$$\begin{aligned}
&\fint_{B_{hR}}|V(Du)-(V(Du))_{hR}|^2dx\\
&\le ch^{-n}\varphi^*\left(\left\|\frac{\varphi'(|F|)F}{|F|}\right\|_{BMO}\right)+ch^\alpha\fint_{B_R}|V(Du)-(V(Du))_R|^2dx.
\end{aligned}$$

By choosing a suitable h and taking the sup over all $R>0$ we obtain

$$\|V(Du)\|^2_{BMO}\le c\varphi^*\left(\left\|\frac{\varphi'(|F|)F}{|F|}\right\|_{BMO}\right)$$

and the result follows. □

5. Higher Integrability for certain quasi-regular systems

Let $A:\Omega\times\mathbb{R}^{nN}\to\mathbb{R}^{nN}$ where $A(x,\xi)$ is a Carathéodory function and Ω is a bounded and open subset of $\mathbb{R}^n$.

Assume, moreover, the following:

(α) $A(x,0)=0$;
(β) $\langle D_\eta A(x,\eta)\xi,\xi\rangle\ge\gamma\ \varphi''(|\eta|)|\xi|^2$;
(γ) $|D_\eta A(x,\eta)|\le\Gamma(1+\varphi''(|\eta|))$,

with $\varphi(t)$ satisfying Assumption 2.2. Here γ and $\Gamma>0$ are positive constants.

We shall need the following theorem.

Theorem 5.1. *Let n, $d\in\mathbb{N}$, $\Omega\subset\mathbb{R}^n$ be a bounded and open set, $\sigma\in(0,1)$, $M>0$, $H\in L^\varphi(\Omega,\mathbb{R}^d)$ be such that for every cube $Q\subset\subset\Omega$ either*

$$\fint_Q\varphi(|H|)dx\le M \tag{35}$$

or

$$\fint_{\sigma Q} \varphi(|H - H_{\sigma Q}|)dx \le \epsilon \fint_Q \varphi(|H|)dx \tag{36}$$

where $0 < \epsilon < \epsilon_0 = \epsilon_0(n, \Delta_2(\varphi))$.

Then there exists a constant $c = c(\epsilon, n, \Delta_2(\varphi))$ *such that for all* $\psi \succ \varphi$ *with* $\Delta_2(\psi) < c$*, it is* $H \in L^{\psi}_{loc}(\Omega, \mathbb{R}^d)$ *and for each* $\Omega' \subset\subset \Omega$

$$\varphi\left(\psi^{-1}\left(\int_{\Omega'} \psi(|H|)dx\right)\right) \le C\rho^{-n} \int_{\Omega} (\varphi(|H|) + M)dx, \tag{37}$$

where $\rho = \mathrm{dist}(\Omega', \partial\Omega)$ *and* C *is a constant dependent only on* σ, n, $\Delta_2(\varphi)$, $\Delta_2(\psi)$, Ω.

The proof of the Theorem 5.1 can be found in [26]; in the power case can be found in [18] and [15].

Let us state the main result of this Section.

Theorem 5.2. *Let* A *be as before and* φ *as in Theorem* 5.1.

Let $p > 2$ *and* $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ *be a weak solution of the system*

$$\mathrm{div}\,(A(x, Du)) = 0. \tag{38}$$

If it holds, for all $\xi \in \mathbb{R}^{nN}$,

$$\left|\frac{\varphi'(|\xi|)\xi}{|\xi|} - A(x,\xi)\right| \le \epsilon\varphi'(|\xi|) \quad a.e.\ x \in \Omega. \tag{39}$$

with $0 < \epsilon < \epsilon_0(n, p, \varphi)$*, then* $V(Du) \in L^p_{loc}(\Omega, \mathbb{R}^{nN})$.

Proof. The proof can be obtained adapting to the systems the arguments used in [26] to prove Theorem 4.2 in the scalar case. Let $B = B_r \subset\subset \Omega$ and let $v \in W^{1,\varphi}(B, \mathbb{R}^N)$ be the solution of the system

$$\begin{cases} \mathrm{div}\, \dfrac{\varphi'(|Dv|)Dv}{|Dv|} = 0 & \text{in } B \\ v = u & \text{on } \partial B. \end{cases} \tag{40}$$

We have, by (14a):

$$\int_B |V(Du) - V(Dv))|^2 dx \le c \int_B \left\langle Du - Dv, \frac{\varphi'(|Du|)Du}{|Du|} - \frac{\varphi'(|Dv|)Dv}{|Dv|}\right\rangle dx.$$

The right hand side of the last inequality coincides with

$$\begin{aligned} &c\int_B \left\langle Du - Dv, \frac{\varphi'(|Du|)Du}{|Du|}\right\rangle dx \\ &= c\int_B \left\langle Du - Dv, \frac{\varphi'(|Du|)Du}{|Du|} - A(x, Du)\right\rangle dx + c\int_B \langle Du - Dv, A(x, Du)\rangle\, dx. \end{aligned}$$

Now, since u is a weak solution of the system in (38), Young's inequality and estimates (39) and (15) imply

$$\begin{aligned}
&\int_B |V(Du) - V(Dv)|^2 dx \\
&\leq c \int_B \left\langle Du - Dv, \frac{\varphi'(|Du|)Du}{|Du|} - A(x, Du) \right\rangle dx \\
&\leq c\,\epsilon \int_B |Du - Dv| \varphi'(|Du|) dx \\
&\leq c\epsilon \int_B \varphi(|Du - Dv|) dx + c\epsilon \int_B \varphi(|Du|) dx \\
&\leq c\epsilon \int_B \varphi_{|Dv|}(|Du - Dv|) dx + c\epsilon \int_B \varphi(|Dv|) dx + c\epsilon \int_B \varphi(|Du|) dx \\
&\leq c\epsilon \int_B |V(Du) - V(Dv)|^2 dx + c\epsilon \int_B \varphi(|Du|) dx
\end{aligned}$$

where we have used (14b). Hence, we obtain

$$⨍_B |V(Du) - V(Dv)|^2 dx \leq c\epsilon ⨍_B \varphi(|Du|) dx \leq c\epsilon ⨍_B |V(Du)|^2 dx \tag{41}$$

Then, for $0 < \sigma < 1$, we have

$$⨍_{\sigma B} |V(Du) - V(Dv)|^2 dx \leq \frac{c\epsilon}{\sigma^n} ⨍_B |V(Du)|^2 dx. \tag{42}$$

Now, let $Q = Q_R \subset\subset \Omega$, $0 < \sigma < \frac{1}{\sqrt{n}}$ so that $\sigma Q \subset B_{(\frac{\sigma}{2}\sqrt{n})R} \subset B_{\frac{R}{2}} \subset Q$, and let v be the solution of the problem (40) in the ball $B = B_{\frac{R}{2}}$. Applying (42), (24) and (23) in the ball $B_{\frac{R}{2}}$, we obtain

$$\begin{aligned}
&⨍_{\sigma Q} |V(Du) - (V(Du))_{\sigma Q}|^2 dx \\
&\leq c ⨍_{\sigma Q} \left| V(Du) - (V(Du))_{B_{(\sigma\sqrt{n}\frac{R}{2})}} \right|^2 dx \\
&\leq c ⨍_{B_{(\sigma\sqrt{n}\frac{R}{2})}} \left| V(Du) - (V(Du))_{B_{(\sigma\sqrt{n}\frac{R}{2})}} \right|^2 dx \\
&\leq c ⨍_{B_{(\sigma\sqrt{n}\frac{R}{2})}} |V(Du) - V(Dv)|^2 dx \\
&\qquad + c ⨍_{B_{(\sigma\sqrt{n}\frac{R}{2})}} \left| V(Dv) - (V(Dv))_{B_{(\sigma\sqrt{n}\frac{R}{2})}} \right|^2 dx \\
&\leq c \left(\frac{\epsilon}{\sigma^n} + \sigma^\alpha \right) ⨍_{B_{\frac{R}{2}}} |V(Du)|^2 dx \\
&\leq c \left(\frac{\epsilon}{\sigma^n} + \sigma^\alpha \right) ⨍_Q |V(Du)|^2 dx.
\end{aligned}$$

The result comes immediately from Theorem 5.1 with $H = V(Du)$ and $\varphi(t) = t^2$, choosing, for example, $\sigma^n = \sqrt{\epsilon}$. □

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