Everywhere regularity of functionals with $\varphi$-growth

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Abstract. We prove $C^{1,\alpha}$-regularity for local minimizers of functionals with $\varphi$-growth, giving also the decay estimate. In particular, we present a unified approach in the case of power-type functions.

1. Introduction

Let $\varphi$ be a convex, $C^1$-function and consider the functional:

$$\mathcal{F}(u) = \int_{\Omega} \varphi(|\nabla u|) \, dx$$

(1.1)

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and $u : \Omega \to \mathbb{R}^N$. The standard examples for convex functions $\varphi$ are

$$\varphi(t) = \int_0^t (\kappa + s^2) \frac{p-2}{2} s \, ds$$

and

$$\varphi(t) = \int_0^t (\kappa + s)^{p-2} s \, ds,$$

where $\kappa \geq 0$.

We say that $u$ is a local minimizer for $F$ if

$$\mathcal{F}(u, \text{spt } v) \leq \mathcal{F}(u + v, \text{spt } v) \quad \forall v \in C^1_0(\Omega).$$

The associated Euler Lagrange system is

$$- \text{div} \left( \varphi'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right) = 0$$

(1.2)

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In a fundamental paper Uhlenbeck [26] proved everywhere $C^{1,\alpha}$-regularity for local minimizers of the $p$-growth functional with $p \geq 2$. Later on a large number of generalizations have been made. The case $1 < p < 2$ was considered by Acerbi and Fusco [1] where also the dependence of the functional from $x$ and $u$ was investigated, (see [23] for a complete list).

Lieberman [18] generalized the regularity theory of Ladyzhenskaya and Uraltseva for equations with $\varphi$-growth. Lipschitz regularity results for systems or functionals with nonstandard growth conditions have been considered by Marcellini [19–21] and Esposito et al. [10,11]. We refer to a recent book of Bildhauer [3] for a general treatment.

In a recent paper [22] Marcellini and Papi proved Lipschitz regularity for local minimizers of functionals with growth conditions general enough to embrace linear and exponential ones. A general approach in order to get $C^{1,\alpha}$-regularity for systems is to prove first Lipschitz continuity and then, using the $C^1$-property of the operator, conclude with the help of classical results. Another approach is contained in a paper of Esposito and Mingione [12] in which they raised the question of proving $C^{1,\alpha}$-regularity of $\varphi$-growth by comparison with powers.

Unfortunately, this is not enough to get an excess decay estimate out of the power case.

For this reason, our goal is to prove the $C^{1,\alpha}$-regularity for functionals with $\varphi$-growth giving the decay estimate of the excess functional:

$$\Phi(u, B) = \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 \, dx$$  \hspace{1cm} (1.3)

where $V(Q) = \sqrt{\varphi(\|Q\|)/\|Q\|} Q$ and $B \subset \Omega$ is a ball. To this aim, we make suitable assumptions on the function $\varphi$ in order to ensure the continuity of the second derivatives of $\varphi$. In particular, the case of slow growth [13] and fast growth are ruled out.

Our main theorem is the following:

**Theorem 1.1.** Let $u \in W^{1,\varphi}_{loc}(\Omega)$ be a local minimizer of (1.1), where $\varphi$ satisfies Assumption 2.2. Then $V(\nabla u)$ and $\nabla u$ are locally $\alpha$-Hölder continuous for some $\alpha > 0$.

We present a unified approach to the superquadratic and subquadratic $p$-growth, also considering more general functions than the powers.

The results presented here rely on some technical lemmas that have been proved in a paper of Diening and Ettwein [6], where they get fractional estimates for non-differentiable elliptic systems with $\varphi$-growth.

### 2. Notation and preliminaries

To simplify the notation, the letter $c$ will denote any positive constant, which may vary throughout the paper. For $w \in L^1_{loc}(\mathbb{R}^n)$ and a ball $B \subset \mathbb{R}^n$ we define

$$\langle w \rangle_B := \int_B w(x) \, dx := \frac{1}{|B|} \int_B w(x) \, dx,$$  \hspace{1cm} (2.1)
where \(|B|\) is the \(n\)-dimensional Lebesgue measure of \(B\). For \(\lambda > 0\) we denote by 
\(\lambda B\) the ball with the center as \(B\) but \(\lambda\)-times the radius. By \(e_1, \ldots, e_n\) we denote \nThe unit vectors of \(\mathbb{R}^n\). For \(U, \Omega \subset \mathbb{R}^n\) we write \(U \Subset \Omega\) if \(U\) is a \compact subset of \(\Omega\). We define \(\delta_{i,j} := 0\) for \(i \neq j\) and \(\delta_{i,i} = 1\).

The following definitions and results are standard in the context of N-functions. A real function \(\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is said to be an N-function if it satisfies the following conditions: \(\varphi(0) = 0\) and there exists the derivative \(\varphi'\) of \(\varphi\). This derivative is right continuous, non-decreasing and satisfies \(\varphi'(0) = 0, \varphi'(t) > 0\) for \(t > 0\), and \(\lim_{t \to \infty} \varphi'(t) = \infty\). Especially, \(\varphi\) is convex.

We say that \(\varphi\) satisfies the \(\Delta_2\)-condition, if there exists \(c_1 > 0\) such that for all \(t \geq 0\) holds \(\varphi(2t) \leq c_1 \varphi(t)\). By \(\Delta_2(\varphi)\) we denote the smallest constant \(c_1\).

Since \(\varphi(t) \leq \varphi(2t)\) the \(\Delta_2\) condition is equivalent to \(\varphi(2t) \sim \varphi(t)\). For a family \(\{\varphi_i\}_{i=1}^n\) of N-functions we define \(\Delta_2(\{\varphi_i\}_{i=1}^n) := \sup_i \Delta_2(\varphi_i)\).

By \(L^\varphi\) and \(W^{1,\varphi}\) we denote the classical Orlicz and Sobolev–Orlicz spaces, i.e. \(f \in L^\varphi\) iff \(\int \varphi(|f|) \, dx < \infty\) and \(f \in W^{1,\varphi}\) iff \(\nabla f \in L^\varphi\). By \(W_0^{1,\varphi}(\Omega)\) we denote the closure of \(C_0^\infty(\Omega)\) in \(W^{1,\varphi}(\Omega)\).

By \((\varphi')^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n\) we denote the function

\[
(\varphi')^{-1}(t) := \sup \{s \in \mathbb{R}^n : \varphi'(s) \leq t\}.
\]

If \(\varphi'\) is strictly increasing then \((\varphi')^{-1}\) is the inverse function of \(\varphi'\). Then \(\varphi^* : \mathbb{R}^n \rightarrow \mathbb{R}^n\) with

\[
\varphi^*(t) := \int_0^t (\varphi')^{-1}(s) \, ds
\]
is again an N-function and \((\varphi^*)'(t) = (\varphi')^{-1}(t)\) for \(t > 0\). It is the complementary function of \(\varphi\). Note that \(\varphi^*(t) = \sup_{s \geq 0} (st - \varphi(s))\) and \((\varphi^*)^* = \varphi\). For all \(\delta > 0\) there exists \(c_\delta\) (only depending on \(\Delta_2(\{\varphi, \varphi^*\})\)) such that for all \(t, s \geq 0\) holds

\[
(ts \leq \delta \varphi(t) + c_\delta \varphi^*(s)).
\] (2.2)

For \(\delta = 1\) we have \(c_\delta = 1\). This inequality is called Young’s inequality. For all \(t \geq 0\)

\[
\frac{t}{2} \varphi'\left(\frac{t}{2}\right) \leq \varphi(t) \leq t \varphi'(t),
\]
(2.3)

\[
\varphi^*\left(\frac{\varphi^*(t)}{t}\right) \leq \varphi^*(t) \leq \varphi^*\left(\frac{2 \varphi^*(t)}{t}\right).
\]

Therefore, uniformly in \(t \geq 0\)

\[
\varphi(t) \sim \varphi'(t)t, \quad \varphi^*\left(\varphi'(t)\right) \sim \varphi(t),
\] (2.4)

where the constants only depend on \(\Delta_2(\{\varphi, \varphi^*\})\). If \(\rho(t) = a \varphi(bt)\) for some \(a, b > 0\) and all \(t \geq 0\), then

\[
\rho^*(t) = a \varphi^*\left(\frac{t}{ab}\right).
\] (2.5)
If \( \varphi \) and \( \rho \) are N-functions with \( \varphi(t) \leq \rho(t) \) for all \( t \geq 0 \), then
\[
\rho^*(t) \leq \varphi^*(t)
\]  
(2.6)
for all \( t \geq 0 \).

Throughout the paper we will assume \( \varphi \) satisfies the following assumption.

**Assumption 2.1.** Let \( \varphi \) be an N-function such that \( \varphi \) is \( C^1 \) on \([0, \infty)\) and \( C^2 \) on \((0, \infty)\). Further assume that
\[
\varphi'(t) \sim t \varphi''(t)
\]  
(2.7)
uniformly in \( t > 0 \).

We remark that under these assumptions \( \Delta_2(\{\varphi, \varphi^*\}) < \infty \) will be automatically satisfied, where \( \Delta_2(\{\varphi, \varphi^*\}) \) depends only on the constant in (2.7). In the proof of the regularity Theorem we will additionally require that \( \varphi'' \) is Hölder continuous away from zero.

**Assumption 2.2.** Let \( \varphi \) be as in Assumption 2.1 such that there exists \( \beta \in (0, 1] \) and \( c > 0 \) such that
\[
|\varphi''(s + t) - \varphi''(t)| \leq c \varphi''(t) \left( \frac{|s|}{t} \right)^\beta
\]  
(2.8)
for all \( t > 0 \) and \( s \in \mathbb{R} \) with \( |s| < \frac{1}{2}t \).

**Remark 2.3.** Let \( \varphi \) satisfy Assumption 2.1. Further, let \( t > 0 \) and \( s \in \mathbb{R} \) with \( |s| < \frac{1}{2}t \). Then by Taylor’s formula, \( |s + t| \sim t \) and \( \Delta_2(\varphi) < \infty \) get
\[
|\varphi'(s + t) - \varphi'(t)| \leq c \varphi''(s + t) |s| \leq c \frac{\varphi'(s + t)}{s + t} |s| \leq c \varphi'(t) \frac{|s|}{t}.
\]
So \( \varphi' \) is Lipschitz continuous away from zero. Compare this with (2.8).

We notice that assumption (2.8) is satisfied for example in all of the following three cases.

\[
\begin{align*}
\varphi(t) &= t^p, \\
\varphi(t) &= t^p \log^\beta(e + t), \quad \beta > 0, \\
\varphi(t) &= t^p \log \log(e + t),
\end{align*}
\]
with \( 1 < p < \infty \).

For given \( \varphi \) we define the associated N-function \( \psi \) by
\[
\psi'(t) := \sqrt{\varphi'(t) t}.
\]  
(2.9)
Note that
\[
\psi''(t) = \frac{1}{2} \left( \frac{\varphi''(t)}{\psi'(t)} t + 1 \right) \sqrt{\frac{\varphi'(t)}{t}} = \frac{1}{2} \left( \frac{\varphi''(t)}{\psi'(t)} t + 1 \right) \frac{\psi'(t)}{t}.
\]  
(2.10)
It is shown in [6, Lemma 25] that if \( \varphi \) satisfies Assumption 2.1, then also \( \varphi^*, \psi, \) and \( \psi^* \) satisfy this assumption and \( \psi''(t) \sim \sqrt{\varphi'(t)} \).

Define \( A, V : \mathbb{R}^{N \times n} \rightarrow \mathbb{R}^{N \times n} \) in the following way:

\[
A(Q) = \varphi'(|Q|) \frac{Q}{|Q|}, \tag{2.11a}
\]

\[
V(Q) = \psi'(|Q|) \frac{Q}{|Q|}. \tag{2.11b}
\]

For \( \lambda \geq 0 \) we define the shifted N-function \( \varphi_\lambda \) by

\[
\varphi_\lambda(t) := \int_0^t \varphi'_\lambda(s) \, ds \tag{2.12}
\]

for \( t > 0 \). The shifted N-functions have been introduced in [6]. In [8] and [7] they have been used in the a priori and a posteriori analysis of finite element approximations of (1.1). See [24] for a detailed study of the shifted N-functions.

The connection between \( A, V, \) and \( \{ \varphi_\lambda \}_{\lambda \geq 0} \) is best reflected in the following lemma.

**Lemma 2.4.** Let \( \varphi \) satisfy Assumption 2.1 and let \( A \) and \( V \) be defined by (2.11). Then

\[
(A(P) - A(Q)) \cdot (P - Q) \sim |V(P) - V(Q)|^2 \sim \varphi_P(|P - Q|), \tag{2.13a}
\]

\[
\sim |P - Q|^2 \varphi''(|P + |Q|), \tag{2.13b}
\]

and

\[
|A(P) - A(Q)| \sim \varphi'_P(|P - Q|) \tag{2.13d}
\]

\[
\sim \varphi''(|P + |Q|) |P - Q| \tag{2.13c}
\]

uniformly in \( P, Q \in \mathbb{R}^{N \times n} \). Moreover,

\[
A(Q) \cdot Q \sim |V(Q)|^2 \sim \varphi(|Q|), \tag{2.13f}
\]

\[
|A(Q)| \sim \varphi'(|Q|) \tag{2.13g}
\]

uniformly in \( Q \in \mathbb{R}^{N \times n} \). Note that if \( \varphi''(0) \) does not exists, the expression in (2.13c) and (2.13e) are continuously extended by zero for \( |P| = |Q| = 0 \).

**Proof.** The lemma is a direct consequence of Lemma 3 and Lemma 21 of [6]. \( \square \)

**Remark 2.5.** By definition of \( \varphi_\lambda \) it follows directly, that \( (\varphi_\lambda)_{\lambda_2} = \varphi_{\lambda + \lambda_2} \) for \( \lambda, \lambda_2 \geq 0 \).

It has been proved in [6] that the functions \( \varphi_\lambda \) with \( \lambda \geq 0 \) share the same properties of \( \varphi \). In particular, we have the following result.
Lemma 2.6. Let $\varphi$ satisfy Assumption 2.1. Then for all $\lambda \geq 0$ the function $\varphi_\lambda$ satisfies Assumption 2.1 and $\Delta_2((\varphi_\lambda)_{\lambda \geq 0}, ((\varphi_\lambda)^*)_{\lambda \geq 0}) < \infty$. Moreover,

$$\varphi''_\lambda(t) \sim \varphi''(\lambda + t) \sim \frac{\varphi'(\lambda + t)}{\lambda + t} = \frac{\varphi'_\lambda(t)}{\lambda + t}$$

(2.14)

uniformly in $\lambda, t \geq 0$ with $\lambda + t > 0$. In particular, $\varphi_\lambda$ satisfies Assumption 2.1 with constants independent of $\lambda \geq 0$. Moreover, for $t > 0$ holds

$$\varphi''_\lambda(t) = \frac{\varphi''(\lambda + t) t}{\lambda + t} + \frac{\varphi'((\lambda + t) \lambda)}{(\lambda + t)^2}.$$ 

(2.15)

If $\lambda > 0$, then $\varphi_\lambda$ is $C^2$ on $[0, \infty)$.

If $\varphi$ satisfies Assumption 2.2, then $\varphi_\lambda$ satisfies Assumption 2.2 with $\beta > 0$ and the constant in (2.8) does not depend on $\lambda$.

Proof. Let $\varphi$ satisfy Assumption 2.1. The formula for $\varphi''_\lambda$ follows directly from the definition of $\varphi_\lambda$. This implies the $C^2$-property on $[0, \infty)$ for $\lambda > 0$. The other claims have been proved in [6, Lemma 24 and 27].

Assume now that $\varphi$ satisfies Assumption 2.2 with $\beta > 0$. Then it follows after a short computation from (2.15), (2.8), and Remark 2.3 that (2.8) holds for all $\varphi_\lambda$ with the same $\beta$ such that the constant does not depend on $\lambda \geq 0$. \qed

We state a generalization of Lemma 2.1 of [1] to the context of convex functions $\varphi$.

Lemma 2.7. [6, Lemma 20] Let $\varphi$ be an N-function with $\Delta_2((\varphi, \varphi^*)) < \infty$. Then uniformly for all $P_0, P_1 \in \mathbb{R}^{N \times n}$ with $|P_0| + |P_1| > 0$ holds

$$\int_0^1 \frac{\varphi'(|P_\theta|)}{|P_\theta|} d\theta \sim \frac{\varphi'(|P_0| + |P_1|)}{|P_0| + |P_1|},$$

(2.16)

where $P_\theta := (1 - \theta)P_0 + \theta P_1$. The constants only depend on $\Delta_2((\varphi, \varphi^*))$.

Remark 2.8. Some of the results in this paper are stated under the condition that $\Delta_2((\varphi, \varphi^*)) < \infty$ or that $\varphi$ satisfies Assumption 2.1. Due to Lemma 2.7 all of these results remain valid for $\varphi_\lambda$ with $\lambda \geq 0$ as well.

The following result is contained in [6, Lemma 31].

Lemma 2.9. Let $\varphi$ be an N-function with $\Delta_2((\varphi, \varphi^*)) < \infty$. Then there exist $q_2 > 1, c > 0$ which only depend on $\Delta_2((\varphi, \varphi^*))$ such that

$$\varphi_\lambda(\theta t) \leq c \theta^{q_2} \varphi_\lambda(t)$$

(2.17)

for all $t, \lambda \geq 0$ and all $\theta \in [0, 1]$. 
Since \((\varphi^*)'\) is the inverse of \(\varphi'\), it follows that

\[
A^{-1}(Q) = (\varphi^*)'(|Q|) \frac{Q}{|Q|}.
\]

(2.18)

for all \(Q \in \mathbb{R}^{N \times n}\). In particular, \(A\) is invertible. The same holds for \(A_\lambda\) and \(V_\lambda\).

The following result generalizes Lemma 3 of [9] to the context of convex functions.

**Lemma 2.10.** Let \(\varphi\) satisfy Assumption 2.1. Then there exists \(\beta > 0\), which only depends on the constant in (2.7), such that \(\varphi', (\varphi^*)', \psi',\) and \((\psi^*)'\) are \(\beta\)-Hölder continuous on \([0, \infty)\) and \(A, A^{-1}, V, \) and \(V^{-1}\) are \(\beta\)-Hölder continuous on \(\mathbb{R}^{N \times n}\).

**Proof.** Let \(q_2\) be as in Lemma 2.9, then \(q_2\) only depends on \(\Delta_2(\{\varphi, \varphi^*\})\). We will show that \(\varphi'\) is \(\beta\)-Hölder continuous, with \(\beta := q_2 - 1\). Let \(a, b \in [0, \infty)\) with \(|a - b| \leq 1\). Then by (2.13d) applied to the case \(n = 1\) we get

\[
|\varphi'(a) - \varphi'(b)| \leq c \varphi_b(|a - b|).
\]

Now, with \(t \varphi'(t) \sim \varphi(t)\) and (2.17) follows

\[
|\varphi'(a) - \varphi'(b)| \leq c \frac{\varphi_b(|a - b|)}{|a - b|} \leq c |a - b|^{q_2 - 1} \varphi_b(1).
\]

This proves that \(\varphi'\) is \(\beta\)-Hölder continuous, where \(\beta > 0\) only depends on \(\Delta_2(\{\varphi, \varphi^*\})\).

Due to [6, Lemma 25] also \(\varphi^*\) and \(\psi\) satisfy Assumption 2.1. Thus, we see that also \((\varphi^*)', \psi',\) and \((\psi^*)'\) are \(\alpha\)-Hölder continuous, where \(\alpha > 0\) only depends on the constant in (2.7). Now, the definition of \(A\) and \(V\) and (2.18) imply that \(A, A^{-1}, V,\) and \(V^{-1}\) are \(\alpha\)-Hölder continuous. \(\square\)

**Remark 2.11.** Due to Lemma 2.6 it is possible to apply Lemma 2.10 also to the shifted-versions uniformly in \(\lambda \geq 0\). For example there exists \(\beta > 0\) such that \(V^{-1}_\lambda\) is \(\beta\)-Hölder continuous for all \(\lambda \geq 0\).

**Remark 2.12.** Let \(\varphi\) satisfy Assumption 2.1. Then, for all \(Q, H \in \mathbb{R}^{N \times n}\) with \(Q \neq 0\) holds

\[
\sum_{ij \alpha \gamma} \partial_{\alpha \gamma} A_{ij}(Q) H_{ij} H_{\alpha \gamma} \sim \varphi''(|Q|) |H|^2.
\]

This follows easily from

\[
\partial_{\alpha \gamma} A_{ij}(Q) = \frac{\varphi'(|Q|)}{|Q|^2} \delta_{i, j} \delta_{\alpha, \gamma} + \left( \varphi''(|Q|) - \frac{\varphi'(|Q|)}{|Q|^2} \right) \frac{Q_{ij} Q_{\alpha \gamma}}{|Q|^2}.
\]

(2.19)

and (2.7).
3. Caccioppoli estimates and a Gehring type result

In this paragraph we recall some estimates obtained in the paper [6] for $\varphi$-systems. Since they are local estimates, they hold true for local minimizers.

If $u$ is a local minimizer of the functional (1.1), then $u = (u_1, \ldots, u_N)$ solves

$$\text{div} (A(\nabla u)) = 0.$$  (3.1)

In other words for all test function $\xi \in C^\infty_0(\Omega)$$$
\int \sum_{j,k} (A_{jk}(\nabla u)) \partial_k \xi_j \, dx = \int \sum_{j,k} (\varphi'(|\nabla u|)) \frac{\partial_k u_j}{|\nabla u|} \partial_k \xi_j \, dx = 0.  \quad (3.2)$$

Since $u \in W^{1,\varphi}(\Omega)$, it follows that (3.2) also holds for $\xi \in W^{1,\varphi}_0(\Omega)$.

**Theorem 3.1.** [6, Theorem 4 and 9] Let $\varphi$ satisfy Assumption 2.1 and let $u$ be a local minimizer of the functional (1.1). Then there exists $K_1 > 0$ such that for all balls $B \subset \Omega$ with $2B \subset \Omega$ holds

$$\int_B \varphi(|\nabla u|) \, dx \leq K_1 \int_{2B} \varphi\left(\frac{|u - \langle u \rangle_{2B}|}{R}\right) \, dx. \quad (3.3)$$

Note that $K_1$ only depends on the constant in (2.7).

Note that similar results as Theorem 3.1 regarding higher integrability have been proved in [5] by Cianchi and Fusco and [4] by Cianchi.

Another important tool in our proof will be the following generalization of the Poincaré inequality.

**Theorem 3.2.** [6, Theorem 7] Let $\varphi$ be an N-function with $\Delta_2(\varphi, \varphi^*) < \infty$. Further, let $B \subset \mathbb{R}^n$ be some ball with diameter $R$. Then there exist $0 < \theta < 1$ and $K > 0$, which only depend on $\Delta_2(\varphi, \varphi^*)$ and $R$, such that for all $v \in W^{1,\varphi}(B)$ holds

$$\int_B \varphi\left(\frac{|v - \langle v \rangle_B|}{R}\right) \, dx \leq K \left(\int_B \varphi(|\nabla v|) \right)^{\frac{1}{\theta}}. \quad (3.4)$$

We also need the following estimate of Gehring type.

**Theorem 3.3.** [6, Theorem 9] Let $\varphi$ be an N-function with $\Delta_2(\varphi, \varphi^*) < \infty$ and let $u$ be a local minimizer of (1.1). Then there exists $q_1 > 1$ and $c > 1$ such that for all balls $B$ with $2B \subset \Omega$ and all $q \in [1, q_1]$ holds

$$\left(\int_B |\nabla (\nabla u)|^q \, dx\right)^{\frac{1}{q}} \leq c \int_{2B} |\nabla (\nabla u)|^2 \, dx \quad (3.5)$$

Especially, we have $\varphi(|\nabla u|) \in L^{q_1}_{\text{loc}}(\Omega)$. The constants $c$ and $q_1$ only depend on $\Delta_2(\varphi, \varphi^*)$.

\[\text{1} \] The original version is stated with $\langle u \rangle_B$ rather than $\langle u \rangle_{2B}$. The proof for $\langle u \rangle_{2B}$ requires no change.
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In the following we will derive an improved version of this theorem. For this we start with a reverse Hölder estimate.

**Lemma 3.4.** Let $\varphi$ satisfy Assumption 2.1 and let $u$ be a local minimizer of the functional (1.1). Then there exist $0 < \theta < 1$ and $K_2 > 0$ such that for all balls $B \subset \Omega$ with $2B \Subset \Omega$ and all $Q \in \mathbb{R}^{N \times n}$ holds

$$\int_B |V(\nabla u) - V(Q)|^2 \, dx \leq K_2 \left( \int_{2B} |V(\nabla u) - V(Q)|^{2\theta} \, dx \right)^{\frac{1}{\theta}}. \quad (3.6)$$

Note that $\theta, K_2$ only depend on the constant in (2.7).

**Proof.** Let $q > 1$ be as in Lemma 2.9 and let $s := \frac{q^2}{q^2 - 1} > 1$. Let $\eta \in C_0^\infty(2B)$ with $\chi_B \leq \eta \leq \chi_{2B}$ and $|\nabla \eta|_\infty \leq c/R$, where $R$ is the radius of $B$. Let $\xi := \eta^s(u - q)$, where $q : \Omega \to \mathbb{R}^n$ is a linear function such that $\nabla q = Q$ and $\int_{2B} u - q \, dx = 0$. We use $\xi$ as a test function for (3.1) and get

$$0 = \int_{2B} (A(\nabla u) - A(Q)) : \nabla (\eta^s(u - q)) \, dx.$$

Using $\nabla (\eta^s(u - q)) = \eta^s(\nabla u - Q) + s \eta^{s-1} \nabla \eta \otimes (u - q)$ (2.13d), and $|\nabla \eta| \leq c/R$ we get

$$\int_{2B} \eta^s(A(\nabla u) - A(Q)) : (\nabla u - Q) \, dx \leq c \int_{2B} \eta^{s-1} \varphi_{|Q|} (|\nabla u - Q|) \frac{|u - q|}{R} \, dx$$

Using Lemma 2.4 and Young’s inequality (2.2) we deduce

$$\int_{2B} \eta^s |V(\nabla u) - V(Q)|^2 \, dx \leq \delta \int_{2B} (\varphi_{|Q|})^s \left( \eta^{s-1} \varphi_{|Q|} (|\nabla u - Q|) \right) \, dx$$

$$+ c\delta \int_{2B} \varphi_{|Q|} \left( \frac{|u - q|}{R} \right) \, dx. \quad (3.7)$$

We estimate the first term on the right-hand side using (2.17), $(s - 1)q_2 = s$ (2.4), and Lemma 2.4 by

$$\delta \int_{2B} (\varphi_{|Q|})^s \left( \eta^{s-1} \varphi_{|Q|} (|\nabla u - Q|) \right) \, dx \leq \delta c \int_{2B} \eta^s \varphi_{|Q|} (|\nabla u - Q|) \, dx$$

$$\leq \delta c \int_{2B} \eta^s |V(\nabla u) - V(Q)|^2 \, dx.$$

So we can absorb this term for small $\delta > 0$ in (3.7) on the left-hand side. The second term on the right-hand side of (3.7) can be estimated with the help of the
Poincaré inequality, (see Theorem 3.2), \( \int_{2B} \nabla u - q \, dx = 0 \), and Lemma 2.4 by
\[
c_\delta \int_{2B} \varphi(\frac{|u - q|}{R}) \, dx \leq c_\delta c \left( \int_{2B} ((\varphi(\|\nabla u - Q\|))^{\theta}) \, dx \right)^{\frac{1}{\theta}}
\]
\[
\leq c_\delta c \left( \int_{2B} |\nabla (\nabla u) - \nabla (Q)|^{2\theta} \, dx \right)^{\frac{1}{\theta}}
\]
where \( \theta \in (0, 1) \) is as in Theorem 3.2. Combining the estimates starting with (3.7) we have proved (3.6).

Note that Lemma 3.4 is an improved version of Theorems 3.1 and 3.2. Indeed, if we combine Theorems 3.1 and 3.2 with the estimate \( \varphi(|\nabla u|) \sim |\nabla (\nabla u)| \) from Lemma 2.4, then we immediately get 3.5 with \( Q = 0 \). Lemma 3.4 is an improvement, since we are allow to substract \( \nabla (Q) \) in the integrals, providing us with a reverse Hölder estimate for the oscillation.

Applying the ingenious lemma of Giaquinta and Modica to Lemma 3.4 we immediately get the following improved version of Theorem 3.3.

**Lemma 3.5.** Let \( \varphi \) satisfy Assumption 2.1 and let \( u \) be a local minimizer of the functional (1.1). Then there exists \( q_3 > 1 \) and \( c > 1 \) such that for all balls \( B \) with \( 2B \subset \Omega \), all \( q \in [1, q_3] \), and all \( Q \in \mathbb{R}^{N \times n} \) holds
\[
\left( \int_B |\nabla (\nabla u) - \nabla (Q)|^{2q} \, dx \right)^{\frac{1}{q}} \leq c \int_{2B} |\nabla (\nabla u) - \nabla (Q)|^2 \, dx \tag{3.8}
\]

Note that \( c \) and \( q_3 \) only depend on the constant in (2.7).

Let us introduce the following notations: For \( x, s \in \mathbb{R}^n \) we define
\[
T_s (x) := x + s, \quad (\tau_s f)(x) := f(x + s) - f(x).
\]

The following result is based on [6, Theorem 11].

**Theorem 3.6.** Let \( \varphi \) satisfy Assumption 2.1 and let \( u \) be a local minimizer of the functional (1.1). Then there exists \( c > 0 \) such that if \( B \subset \Omega \) is a ball with \( 2B \subset \Omega \) and if \( h \in \mathbb{R}^n \setminus \{0\} \) with \( |h| \leq R \), where \( R \) is the radius of \( B \), then
\[
\int_B |\tau_h \nabla (\nabla u)|^2 \, dx \leq c \frac{|h|^2}{R^2} \int_{2B} |\nabla (\nabla u)|^2 \, dx. \tag{3.9}
\]

Passing in Theorem 3.6 to \( h \to 0^+ \) we immediately get the following estimate for the gradients of \( \nabla (\nabla u) \).

**Corollary 3.7.** Let \( \varphi \) satisfy Assumption 2.1 and let \( u \) be a local minimizer of the functional (1.1). Then \( \nabla (\nabla u) \in W^{1,2}_{\text{loc}}(\Omega) \) and there exists \( c > 0 \) such that if \( B \subset \Omega \) is a ball with radius \( R \) and \( 2B \subset \Omega \), then
\[
\int_B |\nabla (\nabla u)|^2 \, dx \leq c \frac{|\nabla (\nabla u)|^2}{R^2} \int_{2B} |\nabla (\nabla u)|^2 \, dx. \tag{3.10}
\]
Corollary 3.8. Let $\varphi$ satisfy Assumption 2.1 and let $u$ be a local minimizer of the functional (1.1). If $n \geq 3$, then there exists $c > 0$ such that if $B \subset \Omega$ is a ball with $2B \Subset \Omega$, then

$$
\left( \int_B |V(\nabla u)|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}} \leq c \int_{2B} |V(\nabla u)|^2 \, dx.
$$

(3.11)

If $n = 2$, then the inequality holds if we replace on the left-hand side $\frac{2n}{n-2}$ and $\frac{n-2}{2n}$ by $q$ and $\frac{1}{q}$, respectively, where $q \in [1, \infty)$. In this case $c = c(q)$. If $n = 1$, then we can use $q = \infty$.

Proof. The result follows from the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ for $n \geq 3$, $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for $n = 2$ and $q \in [1, \infty)$, and $W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $n = 1$. \qed

4. Approximated system

In order to study system (1.1) it is sometimes more convenient to examine an approximated version of the system. For the approximation we use the shifted-N-functions as introduced in (2.12). We will see that $\varphi_\lambda$ is a good approximation of $\varphi$ while it has a better behaviour at zero, see in particular Lemma 2.6. Later we will pass to the limit $\lambda \to 0$ to transfer our results to the original system (1.1).

If not stated otherwise we will assume that $\varphi$ satisfies Assumption 2.1. As an approximation of (1.1) we consider for $\lambda \geq 0$ the functional

$$
\mathcal{F}_\lambda(v) = \int_\Omega \varphi_\lambda(|\nabla v|) \, dx.
$$

(4.1)

In analogy to (2.11) we define $A_\lambda$, $V_\lambda : \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ for $\lambda \geq 0$ by

$$
A_\lambda(Q) = \varphi'_\lambda(|Q|) \frac{Q}{|Q|},
$$

(4.2a)

$$
V_\lambda(Q) = \psi'_\lambda(|Q|) \frac{Q}{|Q|},
$$

(4.2b)

where $\psi_\lambda$ is the associated N-function (compare (2.9)) given by

$$
\psi'_\lambda(t) := \sqrt{\varphi'_\lambda(t) t}.
$$

(4.3)

Since

$$
\psi'_\lambda(t) = \sqrt{\varphi'_\lambda(t) t} = \frac{\varphi'(\lambda + t)}{\lambda + t} = \frac{\psi'(\lambda + t)}{\lambda + t} t
$$

for $\lambda, t > 0$, the function $\psi_\lambda$ is just the shifted version of $\psi$. Therefore, there will be no confusion in the notation.

Since $\varphi_0 = \varphi$, we recover for $\lambda = 0$ in (4.1) our original system (1.1). In particular, $\mathcal{F}_0 = \mathcal{F}$, $A_0 = A$, $V_0 = V$, and $\psi_0 = \psi$. 
Remark 4.1. Let \( \varphi \) be an N-function with \( \Delta_2(\{|\varphi, \varphi^*|\}) < \infty \). The following version of Young’s inequality has been shown in [6, Lemma 32]. It holds

\[
\varphi'_\lambda(t)s \leq c\varphi_\lambda(t) + c\varphi_\lambda(s)
\] (4.4)

for all \( \lambda, s, t \geq 0 \), where \( c \) does only depend on \( \Delta_2(\{|\varphi, \varphi^*|\}) \). It has been shown in [6, Lemma 30] that

\[
\varphi_\lambda(t) \leq c\varphi_\lambda^2(\lambda),
\] (4.5)

for all \( \lambda \geq 0 \) and \( \theta \in [0, 1] \), where \( c \) does only depend on \( \Delta_2(\{|\varphi, \varphi^*|\}) \). Moreover, it has been shown in [7, Lemma 25 + Corollary 26] that

\[
\varphi_\lambda^2(t) \leq c\varphi_\lambda(t) + c\varphi_\lambda^2(s)
\] (4.6)

for all \( Q, P \in \mathbb{R}^{N \times n} \) and all \( t \geq 0 \) and

\[
\varphi_\lambda(t) \leq c(\varphi(t) + \varphi(\lambda)),
\] (4.7)

for all \( t, \lambda \geq 0 \), where \( c \) does only depend on \( \Delta_2(\{|\varphi, \varphi^*|\}) \).

In particular, (4.7) implies that \( L^\varphi(\Omega) \hookrightarrow L^s_{\text{loc}}(\Omega) \) and \( W^{1, \varphi}_\text{loc}(\Omega) = W^{1, \varphi_\lambda}_\text{loc}(\Omega) \) for all \( \lambda \geq 0 \). So for all local results we can still work within the scope of the spaces \( L^\varphi \) and \( W^{1, \varphi} \).

Moreover, it has been shown in [6, Lemma 26] that

\[
(\varphi_\lambda)^*(t) \sim (\varphi^*)_{\varphi_\lambda'(\lambda)}(t),
\] (4.8a)

\[
((\varphi_\lambda)^*)'(t) \sim (\varphi^*)_{\varphi_\lambda'(\lambda)}'(t)
\] (4.8b)

uniformly in \( \lambda, t \geq 0 \).

Lemma 4.2. Let \( \varphi \) be an N-function with \( \Delta_2(\{|\varphi, \varphi^*|\}) < \infty \). Then there exists \( s_0 > 1 \) such that \( L^\varphi(\Omega) \hookrightarrow L^{s_0}_{\text{loc}}(\Omega) \). Note that \( s_0 \) only depends on \( \Delta_2(\{|\varphi, \varphi^*|\}) \).

Proof. Since \( \Delta_2(\varphi^*) < \infty \) it follows from [17, Lemma 1.2.2+1.2.3] that \( \varphi^\theta \) is quasiconvex for some \( 1 - \frac{1}{n} < \theta < 1 \), i.e. there exists an N-function \( \rho \) with \( \varphi^\theta \sim \rho \) and \( \Delta_2((\rho, \rho^*))) < \infty \). It is important to remark that \( \theta \) and \( \Delta_2((\rho, \rho^*))) \) only depend on \( \Delta_2(\{|\varphi, \varphi^*|\})\). Let \( s := \frac{1}{\theta} \), then \( s > 1 \). For \( t \leq 1 \) we have \( t^\theta \leq 1 \). On the other hand for \( t \geq 1 \), we have by the quasiconvexity of \( \varphi^\theta \) that \( \varphi(t) = (\varphi^\theta(t))^\frac{1}{\theta} \geq (c t \varphi^\theta(1))^\frac{1}{\theta} = c t^\theta \varphi(1) \). Overall, we have \( t^\theta \leq 1 + c \varphi(t)/\varphi(1) \) for all \( t \geq 0 \). This proves the claim. \( \square \)

Due to Lemma 2.6 the result above also holds with \( \varphi \) replaced by \( \varphi_\lambda \), where \( s_0 \) is independent of \( \lambda > 0 \).

Lemma 4.3. Let \( \varphi \) satisfy Assumption 2.1. Then there exists \( s_1 > 1 \) such that the following holds. If \( \lambda > 0 \) and \( u_\lambda \) is a local minimizer of the functional (4.1) then \( u_\lambda \in W^{2,s_1}_{\text{loc}}(\Omega) \).
Proof. Let $B$ be a ball with radius $R$ and $2B \Subset \Omega$. We will show that $u_\lambda \in W^{2, s}(B)$ for some $s > 1$, independent of $B$ and $\lambda > 0$. Let $s_0 > 1$ be such that $L^\varphi(\Omega) \hookrightarrow L^{s_0}(2B)$. Choose $s \in (1, 2)$ such that $\frac{s}{2 - s} \leq s_0$. Let $h \in \mathbb{R}^n \setminus \{0\}$ with $|h| \leq R$. With Hölder’s inequality with $\frac{2}{s_0} = \frac{2}{\lambda}$ and $\frac{2}{s}$ we estimate
\[
\left( |h|^{-1} |\tau_h \nabla u_\lambda| \right)^s \leq \left( \varphi''(\lambda) (|\nabla u_\lambda| + |\tau_h \nabla u_\lambda|) \right) \frac{2}{s_0}\frac{s}{s_0} + \varphi''(\lambda)(|\nabla u_\lambda| + |\tau_h \nabla u_\lambda|)|h|^{-2}|\tau_h \nabla u_\lambda|^2
\]
\[=: (I) + (II).
\]
We will show that $(I)$ and $(II)$ are integrable over $B$ with bounds independent of $h$. With Lemma 2.4 we estimate $(II) \leq c |h|^{-2} (|\tau_h \nabla u_\lambda|)^2$. So with Theorem 3.6 we see that $(II)$ is integrable over $B$ with bound independent of $h$. Let us now consider $(I)$. For $t \geq 0$ we estimate with (2.7)
\[
\frac{1}{\varphi''(\lambda)} \leq c \frac{\lambda + t}{\varphi''(\lambda)} \leq \frac{\lambda + t}{\varphi'(\lambda)}.
\]
If we replace $t$ by $|\nabla u_\lambda| + |\tau_h \nabla u_\lambda|$, integrate over $B$ and use $L^\varphi(\Omega) \hookrightarrow L^{s_0}(2B) \hookrightarrow L^{\frac{2}{s_0}}(2B)$, then we see that $(I)$ is integrable over $B$ with bound independent of $h$.
We have shown that
\[
\int_B \left( |h|^{-1} |\tau_h \nabla u_\lambda| \right)^s dx \leq c,
\]
where $c$ depends on $\lambda$ and $u_\lambda$, but is independent of $h$. It follows that $u_\lambda \in W^{2, s}(B)$. \hfill $\square$

Lemma 4.4. Let $\varphi$ satisfy Assumption 2.1 and be $C^2$ on $[0, \infty)$. Let $B \subset \Omega$ be a ball with radius $R$ and $2B \Subset \Omega$ and let $w \in W^{2, s}(2B)$ for some $s > 1$ such that $V(\nabla w) \in W^{1, 2}(2B)$. Then
\[
\varphi''(\nabla w)|\partial_i \nabla w|^2 \sim |\partial_i V(\nabla w)|^2
\]
almost everywhere with $i = 1, \ldots, n$. In particular,
\[
\int_B \varphi''(\nabla w)|\partial_i \nabla w|^2 dx \sim \int_B |\partial_i V(\nabla w)|^2 dx.
\]

Proof. Let $r \in (0, R)$. Then by Lemma 2.4 (applied to $V$ and $\psi$) and $\psi''(t) \sim \sqrt{\varphi''(t)}$ it follows that
\[
\varphi''(\nabla w) + |\tau_{r \ell_i} \nabla w|^2 \sim |\partial_{r \ell_i} V(\nabla w)|^2.
\]
Since $V \in W^{1, 2}(B)$, we have $r^{-1}|\tau_{r \ell_i} V(\nabla w) \rightarrow \partial_{r \ell_i} V(\nabla w) \in L^2(B)$ for $r \to 0$ and $r^{-2}|\tau_{r \ell_i} V(\nabla w)|^2 \rightarrow |\partial_{r \ell_i} V(\nabla w)|^2$ in $L^1(B)$. So the right hand side of (4.9) converges in $L^1(B)$ to $|\partial_i V(\nabla w)|^2$. It remains to prove that
\[
\varphi''(\nabla w) + |\tau_{r \ell_i} \nabla w|^2 \sim \varphi''(\nabla w)|\partial_i \nabla w|^2 \quad \text{a.e. in } B,
\]
since then the theorem of dominated convergence proves the claim. Now, \( w \in W^{2,s}(2B) \), so for a subsequence \( r_k \to 0 \) we have \( \tau_{r_k \epsilon_l} \nabla w \to 0 \) and \( r_k^{-1} \tau_{r_k \epsilon_l} \nabla w \to \partial_t \nabla w \) almost everywhere in \( B \). Now, the continuity of \( \varphi'' \) on \([0, \infty)\) proves the claim. \( \square \)

**Lemma 4.5.** Let \( \varphi \) satisfy Assumption 2.1. If \( \lambda > 0 \) and \( u_\lambda \) is a local minimizer of the functional \((4.1)\), \( B \) is a ball with radius \( R \) and \( 2B \subseteq \Omega \), and \( \nabla (\nabla u_\lambda) \in L^{2q}(2B) \) with \( q \geq 1 \), then \( \varphi_\lambda(|\nabla u_\lambda|) \in W^{1,\frac{2q}{2q + 1}}(B) \), \( \varphi_\lambda(|\nabla u_\lambda|)|\nabla^2 u_\lambda| \in L^{\frac{2q}{2q + 1}}(B) \) and

\[
\left( \int_B \nabla \varphi_\lambda(|\nabla u_\lambda|) \frac{2q}{2q + 1} \right)^{\frac{1 + q}{2q}} \leq c \left( \int_{2B} \varphi_\lambda(|\nabla u_\lambda|)|\nabla^2 u_\lambda| \right)^{\frac{2q}{2q + 1}} \leq c R^{-1} \left( \int_{2B} \nabla (\nabla u_\lambda)|^{2q} \right)^{\frac{1}{q}}.
\]

Moreover,

\[
\partial_j \varphi_\lambda(|\nabla u_\lambda|) = \varphi_\lambda'(|\nabla u_\lambda|) \partial_j |\nabla u_\lambda|
\]

for \( j = 1, \ldots, n \).

**Proof.** Let \( \lambda > 0 \). Then by Lemma 4.3 we have \( u_\lambda \in W^{2,s_1}_{\text{loc}}(2B) \) and therefore \( |\nabla u_\lambda| \in W^{1,s_1}_{\text{loc}}(2B) \). So \( |\nabla u_\lambda| \) is absolutely continuous on almost every line (parallel to the coordinate axes). Since \( \varphi_\lambda \) is Lipschitz on \([0, \infty)\), we get that

\[
|\nabla \varphi_\lambda(|\nabla u_\lambda|)| \leq \varphi_\lambda'(|\nabla u_\lambda|)|\nabla^2 u_\lambda| \tag{4.11}
\]
on almost every line. This, Young’s inequality with \( q + 1 \) and \( \frac{q + 1}{q} \), and \((2.7)\) imply

\[
|\nabla \varphi_\lambda(|\nabla u_\lambda|)| \frac{2q}{2q + 1} \leq \left( \varphi_\lambda'(|\nabla u_\lambda|)|\nabla^2 u_\lambda| \right)^{\frac{2q}{2q + 1}} \leq c \left( \sqrt{\varphi_\lambda(|\nabla u_\lambda|)} \varphi''_\lambda(|\nabla u_\lambda|)|\nabla^2 u_\lambda| \right)^{\frac{2q}{2q + 1}} = c \left( t_0 \varphi_\lambda(|\nabla u_\lambda|) \right)^q t_0^{-1} \left( \varphi''_\lambda(|\nabla u_\lambda|)|\nabla^2 u_\lambda| \right)^{\frac{2q}{2q + 1}} \leq c \varphi_\lambda(|\nabla u_\lambda|)|\nabla^2 u_\lambda|^{2q} t_0^{-2}
\]
on almost every line, where \( t_0 > 0 \) will be chosen later. From Lemma 4.3 we know that \( u_\lambda \in W^{2,s_1}_{\text{loc}}(2B) \). So with Lemma 4.4 and Corollary 3.7 we deduce that

\[
\int_B \varphi_\lambda''(|\nabla u_\lambda|)|\nabla^2 u_\lambda|^2 \leq c \int_B |\nabla \varphi_\lambda(|\nabla u_\lambda|)| \leq c R^{-2} \int_{2B} |\nabla \varphi_\lambda(|\nabla u_\lambda|)| \leq \int_{2B} \nabla \varphi_\lambda(|\nabla u_\lambda|)| \leq c R^{-1} \int_{2B} |\nabla \varphi_\lambda(|\nabla u_\lambda|)|^{2q} \leq c \int_B |\nabla \varphi_\lambda(|\nabla u_\lambda|)| \leq c R^{-2} \int_{2B} |\nabla \varphi_\lambda(|\nabla u_\lambda|)|^{2q} \leq c \int_B |\nabla \varphi_\lambda(|\nabla u_\lambda|)| \leq c R^{-1} \int_{2B} |\nabla \varphi_\lambda(|\nabla u_\lambda|)|^{2q} \leq c \int_B \nabla \varphi_\lambda(|\nabla u_\lambda|)|.
\]

On the other hand with Lemma 2.4 we have

\[
\int_B (\varphi_\lambda(|\nabla u_\lambda|))^q dx \leq c \int_B |\nabla \varphi_\lambda(|\nabla u_\lambda|)|^{2q} dx.
\]
Overall, we have shown that
\[
\int_B \left| \nabla \varphi_{\lambda} (|\nabla u_{\lambda}|) \right|^{\frac{2q}{1+q}} \leq t_0^{2q} c \int_B |\nabla \lambda (\nabla u_{\lambda})|^{2q} \, dx + c t_0^{-2} R^{-2} \int_{2B} |\nabla \lambda (\nabla u_{\lambda})|^{2} \, dx
\]
\[
\leq t_0^{2q} c \int_{2B} |\nabla \lambda (\nabla u_{\lambda})|^{2q} \, dx + c t_0^{-2} R^{-2} \left( \int_{2B} |\nabla \lambda (\nabla u_{\lambda})|^{2q} \, dx \right)^{1/q}.
\]
Minimizing over \( t_0 > 0 \) proves
\[
\int_B \left| \nabla \varphi_{\lambda} (|\nabla u_{\lambda}|) \right|^{\frac{2q}{1+q}} \leq c R^{-2} \left( \int_{2B} |\nabla \lambda (\nabla u_{\lambda})|^{2q} \, dx \right)^{\frac{2}{1+q}}.
\]
This proves the claim. \( \square \)

Note that Theorem 3.3 and Lemma 2.6 ensures that the requirements of Lemma 4.5 are always satisfied for some \( q > 1 \), where \( q \) is independent of \( \lambda > 0 \).

**Corollary 4.6.** Let \( \varphi \) satisfy Assumption 2.1. Let \( \lambda > 0 \), let \( u_{\lambda} \) be a local minimizer of the functional \((4.1)\), and let \( B \) be a ball with radius \( R \) and \( 2B \subset \Omega \). For \( n \geq 3 \) we have \( \varphi_{\lambda} (|\nabla u_{\lambda}|) \in W^{1, \frac{n}{n-1}} (B) \) and
\[
\left( \int_B \left| \nabla \varphi_{\lambda} (|\nabla u_{\lambda}|) \right|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq c R^{-1} \int_{2B} |\nabla \lambda (\nabla u_{\lambda})|^{2} \, dx. \quad (4.12)
\]
The constant does not depend on \( \lambda > 0 \). If \( n = 1 \) or \( n = 2 \), then for all \( s \in [1, 2) \) holds \( \varphi_{\lambda} (|\nabla u_{\lambda}|) \in W^{1, s} (B) \) and
\[
\left( \int_B \left| \nabla \varphi_{\lambda} (|\nabla u_{\lambda}|) \right|^s \right)^{\frac{1}{s}} \leq c R^{-1} \int_{2B} |\nabla \lambda (\nabla u_{\lambda})|^{2} \, dx.
\]
The constant does not depend on \( \lambda > 0 \).

**Proof:** The result immediately follows from Lemma 4.5, Corollary 3.7, and Corollary 3.8. \( \square \)

We need the following auxiliary results.

**Lemma 4.7.** Let \( \varphi \) satisfy Assumption 2.1. Further, let \( a > 0 \) and \( U \subset [0, a] \times \mathbb{R}^{N \times n} \). Then \( \sup_{(\lambda, Q) \in U} |Q| < \infty \) if and only if \( \sup_{(\lambda, Q) \in U} |A_{\lambda} (Q)| < \infty \). Moreover, \( \sup_{(\lambda, Q) \in U} |Q| < \infty \) if and only if \( \sup_{(\lambda, Q) \in U} |V_{\lambda} (Q)| < \infty \).
Proof. It suffices to prove the result for $A_\lambda$, since the result for $V_\lambda$ is the same with $\varphi$ replaced by $\psi$. Let $\sup_{(\lambda, Q) \in U} |Q| < \infty$, then by (4.7)

$$\sup_{(\lambda, Q) \in U} |A_{\lambda}(Q)| = \sup_{(\lambda, Q) \in U} \varphi'_\lambda(|Q|) \leq \sup_{(\lambda, Q) \in U} c \left( \varphi'(|Q|) + \varphi'(\lambda) \right) < \infty.$$ 

Assume now that $\sup_{(\lambda, Q) \in U} |A_{\lambda}(Q)| < \infty$. With $A_{\lambda}^{-1}(Q) = ((\varphi_\lambda)^*)'(|Q|) \frac{Q}{|Q|}$, (4.8), and (4.7) (applied to $\varphi^*$) it follows that

$$\sup_{(\lambda, Q) \in U} |Q| = \sup_{(\lambda, Q) \in U} |A_{\lambda}^{-1}(A_{\lambda}(Q))| = \sup_{(\lambda, Q) \in U} ((\varphi_\lambda)^*)'(|A_{\lambda}(Q)|)$$

$$\leq c \sup_{(\lambda, Q) \in U} (\varphi^*)'(\varphi(\lambda) (|A_{\lambda}(Q)|))$$

$$\leq c \sup_{(\lambda, Q) \in U} \left( (\varphi^*)'(|A_{\lambda}(Q)|) + (\varphi^*)'(\varphi(\lambda)) \right)$$

$$= c \sup_{(\lambda, Q) \in U} (\varphi^*)'(|A_{\lambda}(Q)|) + \lambda < \infty.$$ 

This proves the assertion. \box

**Lemma 4.8.** Let $\varphi$ satisfy Assumption 2.1. Then the $(\lambda, Q) \mapsto A_{\lambda}(Q)$, $(\lambda, Q) \mapsto A_{\lambda}^{-1}(Q)$, $(\lambda, Q) \mapsto V_\lambda(Q)$, and $(\lambda, Q) \mapsto V_\lambda^{-1}(Q)$ are continuous on $[0, \infty) \times \mathbb{R}^{N \times n}$.

Proof. Let $(\lambda_k, Q_k) \to (\lambda, Q)$ for $k \to \infty$. If $\lambda + |Q| > 0$, then

$$A_{\lambda_k}(Q_k) = \frac{\varphi'(\lambda_k + |Q_k|)}{\lambda_k + |Q_k|} Q_k \to \frac{\varphi'(\lambda + |Q|)}{\lambda + |Q|} Q = A_{\lambda}(Q).$$

If $\lambda + |Q| = 0$, then $\lambda = 0$, $Q = 0$, and $A_0(Q) = 0$. Moreover,

$$|A_{\lambda_k}(Q_k)| = \frac{\varphi'(\lambda_k + |Q_k|)}{\lambda_k + |Q_k|} |Q_k| \leq \varphi'(\lambda_k + |Q_k|) \to 0.$$ 

So $A_{\lambda_k}(Q_k) \to A_{\lambda}(Q)$ also in this case. This proves that $(\lambda, Q) \mapsto A_{\lambda}(Q)$ is continuous.

We will now show the continuity of $(\lambda, Q) \mapsto A_{\lambda}^{-1}(Q)$. Let $(\lambda_k, Q_k) \to (\lambda, Q)$. We set $P_k := A_{\lambda_k}^{-1}(Q_k)$ and $P := A_{\lambda}^{-1}(Q)$. We have to show $P_k \to P$. By Lemma 4.7 it follows that $P_k$ is bounded. Thus there exists a subsequence $P_{k_j}$ that convergences to some $\overline{P} \in \mathbb{R}^{N \times n}$. Now the continuity of $(\lambda, Q) \mapsto A_{\lambda}(Q)$ implies $Q_{k_j} = A_{\lambda_{k_j}}(P_{k_j}) \to A_{\lambda}(\overline{P})$ for $j \to \infty$. Since $Q_k \to Q$ it follows that $A_{\lambda}(P) = Q = A_{\lambda}(\overline{P})$. This implies $P = \overline{P}$ and therefore $P_{k_j} \to P$. Since the argument works for any subsequence of $P_k$ it follows that the whole sequence $P_k$ converges to $P$. This proves the assertion.

The claim for $V_\lambda$ and $V_\lambda^{-1}$ follows by replacing $\varphi$ by $\psi$ and the remark after (4.3). \box

In the following we will show how to transfer results for the approximated system (4.1) back to our original system (1.1). We will do it only for Corollary 4.6.
Theorem 4.9. Let \( \varphi \) satisfy Assumption 2.1 and let \( u \) be a local minimizer of the functional (1.1). Then Corollary 4.6 holds with \( \varphi_\lambda \) replaced by \( \varphi \).

Proof. We consider the case \( n \geq 3 \). The other case follows analogously. By a simple covering argument, it suffices to prove the result for balls \( B \) with \( 4B \subseteq \Omega \). For \( \lambda > 0 \) let \( u_\lambda \) be the minimizer of

\[
F_\lambda(w) = \int_{4B} \varphi_\lambda(|\nabla w|) \, dx
\]

(4.13)

within the class \( \{ w \in W^{1,\varphi}(4B) : w = u \text{ on } \partial(4B) \} \). Using the properties of \( \varphi \) it is standard to see that such a minimizer exists and is unique. We will show that \( u_\lambda \) converges for \( \lambda \to 0 \) to \( u \). We proceed similarly as in [2].

By (4.7) and Corollary 3.7 we know \( u_\lambda \) is equibounded in \( W^{1,\varphi}(2B) \) and \( V_\lambda(\nabla u_\lambda) \) is equibounded in \( W^{1,2}(2B) \) for \( \lambda \to 0 \). So we have (passing to a subsequence)

\[
u_\lambda \rightharpoonup v \quad \text{in } W^{1,\varphi}(2B),
\]

\[
V_\lambda(\nabla u_\lambda) \rightharpoonup H \quad \text{in } W^{1,2}(2B),
\]

\[
V_\lambda(\nabla u_\lambda) \to H \quad \text{in } L^2(2B)
\]

for some \( v \in W^{1,\varphi}(2B) \) and \( H \in W^{1,2}(2B) \). Passing to a subsequence we have

\[
V_\lambda(\nabla u_\lambda) \to H \quad \text{almost everywhere in } 2B.
\]

So by Lemma 4.8 we deduce that \( \nabla u_\lambda \to V^{-1}(H) \) almost everywhere in \( 2B \). Since the weak limit coincides with the pointwise limit, we get \( V^{-1}(H) = \nabla v \) and \( H = V(\nabla v) \). Since by Corollary 4.6 \( \varphi_\lambda(|\nabla u_\lambda|) \) is equibounded in \( W^{1,\frac{n}{n-1}}(2B) \), there holds \( \varphi_\lambda(|\nabla u_\lambda|) \rightharpoonup g \) in \( W^{1,\frac{n}{n-1}}(2B) \) for a subsequence. Since \( \nabla u_\lambda \to \nabla v \) almost everywhere, we get \( \varphi_\lambda(|\nabla u_\lambda|) \rightharpoonup \varphi(|\nabla v|) \) in \( W^{1,\frac{n}{n-1}}(2B) \). This is enough to pass in (4.12) (for our subsequence) to the limit \( \lambda \to 0 \) and we see that (4.12) holds with \( \varphi_\lambda \) and \( u_\lambda \) replaced by \( \varphi \) and \( v \), respectively. The uniqueness of the minimizer of (4.13) for \( \lambda = 0 \) implies \( u = v \). \( \square \)

Section 5. Subsolution

As in the paper by Uhlenbeck [26] and Acerbi and Fusco [1] we prove that the nonlinear quantity \( \varphi(|\nabla u|) \) is a subsolution of an uniformly elliptic system. As a first step, we will show this for the approximated system (4.1). We proceed similarly to [1].

Lemma 5.1. Let \( \varphi \) be an N-function with \( \Delta_2(\varphi) < \infty \). Then there exists \( s > 1 \) such that

\[
\left( \frac{\varphi(t)}{t} \right)^s \leq 2 \left( \varphi(1) \right)^{s-1} \varphi(t) + \left( \varphi(1) \right)^s
\]

(5.1)

uniformly for all \( t > 0 \). Note that \( s > 1 \) only depends on \( \Delta_2(\varphi) \).
Proof. Let $K \geq 2$ denote the $\Delta_2$-constant of $\varphi$. Then there exists $k \in \mathbb{N}$ with $K \leq 2^k$. Define $s := 1 + \frac{1}{k}$.

If $t \in (0, 1]$, then by convexity $\varphi(t)/t \leq \varphi(1)$, which proves (5.1). Assume now, that $t > 1$. Choose $m \in \mathbb{N}$ such that $2^{m-1} < t \leq 2^m$, then

$$
\left( \varphi(t) \right)^{\frac{s}{t}} \leq \left( \varphi(t) \right)^{\frac{1}{2}} \varphi(t) \leq \left( \varphi(2^m) \right)^{\frac{1}{2}} \varphi(t) \leq K^{\frac{m}{2}} \left( \varphi(1) \right)^{\frac{1}{2}} \varphi(t)
$$

This implies for $t > 1$ that

$$
\left( \frac{\varphi(t)}{t} \right)^s \leq \varphi(t) \frac{t^{1-s}}{(\varphi(1))^{\frac{1}{s}}} \leq 2 \left( \varphi(1) \right)^{s-1} \varphi(t).
$$

This proves the claim. □

Using (2.7), the convexity of $\varphi$, and $\Delta_2(\varphi) < \infty$ we have

$$
\varphi''(\lambda + t) \leq c \varphi''(\lambda + t) \leq c \frac{\varphi(\lambda + t)}{(\lambda + t)^2} \leq c \lambda^{-2} \varphi(\lambda + t) \leq c \lambda^{-2} (\varphi(\lambda) + \varphi(t))
$$

(5.2)

uniformly in $\lambda > 0$ and $t \geq 0$.

Lemma 5.2. Let $\varphi$ satisfy Assumption 2.1. Then there exists $s_2 > 1$ such that if $\lambda > 0$ and $u_\lambda$ is a local minimizer of the functional (4.1), then $A_\lambda(\nabla u_\lambda) \in W^{1,s_2}_{\text{loc}}(\Omega)$.

Proof. Choose $s > 1$ as in Lemma 5.1 and let $s_2 := 2s/(1 + s) \in (1, 2)$, then $s = s_2/(2 - s_2)$ and $1 < s_2 < s$. Let $B \subset \Omega$ be a ball with radius $R$ such that $2B \subset \Omega$. Then $u_\lambda \in W^{1,\varphi_2}(2B)$ and $V_\lambda(\nabla u_\lambda) \in W^{1,2}_1(2B)$. With Lemma 2.4 and Lemma 5.1 we estimate

$$
|A_\lambda(\nabla u_\lambda)|^s \leq c (\varphi'_s(|\nabla u_\lambda|))^s \leq c \left( \varphi_s(\nabla u_\lambda) \right)^s
$$

$$
\leq 2 \left( \varphi_s(1) \right)^{s-1} \varphi_s(\nabla u_\lambda) + c \left( \varphi_s(1) \right)^s.
$$

Since $u_\lambda \in W^{1,\varphi_2}(B)$, it follows from the estimate above that $A_\lambda(\nabla u_\lambda) \in L^s(B)$.

Since $s_2 < s$, we also get $A_\lambda(\nabla u_\lambda) \in L^{s_2}(B)$.

Let $h \in \mathbb{R}^n \setminus \{0\}$ with $|h| < R$. Then

$$
\int_B |h|^{-2} |\tau_h V_\lambda(\nabla u_\lambda)|^2 \, dx \leq \int_{2B} |\nabla (V_\lambda(\nabla u_\lambda))|^2 \, dx.
$$

(5.3)

With Lemma (2.4) and Young’s inequality, with $q = 2/s_2$ and $q' = 2/(2 - s)$, we estimate

$$
\left( |h|^{-1} |\tau_h A_\lambda(\nabla u_\lambda)| \right)^{s_2} \leq c \left( \varphi''_s(\nabla u_\lambda) + |\tau_h \nabla u_\lambda| \right)^{s_2}
$$

$$
\leq \left( \varphi''_s(\nabla u_\lambda) + |\tau_h \nabla u_\lambda| \right)^s
$$

$$
+ \varphi''_s(\nabla u_\lambda) |\tau_h \nabla u_\lambda|^2 \leq \left( \varphi''_s(\nabla u_\lambda) + |\tau_h \nabla u_\lambda| \right)^s + c |h|^{-2} |\tau_h V(\nabla u_\lambda)|^2
$$

$$
=: (I) + (II).
$$
Due to (5.3) the term (II) is in $L^1(B)$ with bound independently of $h$. With $\varphi''(t) \sim \varphi(\lambda + t)/((\lambda + t)^2$ we estimate (I) by Lemma 5.1 as follows:

$$
(I) \leq c \left( \frac{\varphi(\lambda + |\nabla u_\lambda| + |\tau h \nabla u_\lambda|)}{(\lambda + |\nabla u_\lambda| + |\tau h \nabla u_\lambda|)^2} \right)^s
$$

$$
\leq c \lambda^{-s} \left( \frac{\varphi(\lambda + |\nabla u_\lambda| + |\tau h \nabla u_\lambda|)}{(\lambda + |\nabla u_\lambda| + |\tau h \nabla u_\lambda|)^2} \right)^s
$$

$$
\leq c(\varphi(1), s) \lambda^{-s}(\varphi(\lambda + |\nabla u_\lambda| + |\tau h \nabla u_\lambda|) + c(\varphi(1), s)\lambda^{-s}.
$$

The convexity of $\varphi$ and $\Delta_2(\varphi) < \infty$ imply that

$$
(I) \leq c(\varphi(1), s) \lambda^{-s}(\varphi(\lambda) + \varphi(|\nabla u_\lambda|) + \varphi(|\nabla u_\lambda(h + \cdot)|)) + c(\varphi(1), s)\lambda^{-s}.
$$

Since $u_\lambda \in W^{1,\varphi}(2B)$, we get that (I) is in $L^1(B)$ with bound independently of $h$.

Overall, we have shown that $|h|^{-1} \tau h A_\lambda(\nabla u_\lambda)$ is in $L^s(B)$ with bound independent of $h$. Thus $A_\lambda(\nabla u_\lambda) \in W^{1,s}(B)$.

**Lemma 5.3.** Let $\varphi$ satisfy Assumption 2.1 and let $\lambda > 0$. Let $G : \mathbb{R}^n \to \mathbb{R}^{N \times n}$ satisfy $G \in W^{1,s}(B)$ and $A_\lambda(G) \in W^{1,s}(B)$ for some $s > 1$. Then almost everywhere on $B$

$$
\partial_t A^{ij}_{\lambda}(G) = \frac{\varphi''(|G|)}{|G|} \partial_t G_{j k} + \left( \frac{\varphi''(|G|)}{|G|} - \frac{\varphi'(|G|)}{|G|} \right) G_{j k} \partial_t |G|.
$$

**Proof.** Since $G$ and $A_\lambda(G)$ are in $W^{1,s}(B)$, they are absolutely continuous on almost every line (parallel to the coordinate axes). Due to Lemma 2.6, we know that $Q \mapsto A_\lambda(Q)$ is $C^1$ on $\mathbb{R}^{N \times n}$. So (5.4) follows immediately on almost every line.

Let $\varphi$ satisfy Assumption 2.1. For $\lambda > 0$ and $t \geq 0$ we define

$$
\omega_\lambda(t) := \frac{\varphi''(t)}{\varphi'(t)} \left( t - \varphi'(t) \right).
$$

Then by (2.7) and (2.14) it follows that there exists $c_0, c_1 > 0$ such that

$$
c_0 - 1 \leq \omega_\lambda(t) \leq c_1.
$$

for all $t \geq 0$ and all $\lambda > 0$. Note that $c_0$ and $c_1$ only depend on the constant in (2.7).

The next lemma shows that $\varphi_\lambda(|\nabla u_\lambda|)$ is a subsolution to a uniformly elliptic problem, where the constants of ellipticity do no depend on $\lambda > 0$.

**Lemma 5.4.** Let $\varphi$ satisfy Assumption 2.1. Let $\lambda > 0$, let $u_\lambda$ be a local minimizer of the functional (4.1), and let $B$ be a ball with $2B \subseteq \Omega$. Then there exists $G_\lambda : 2B \to \mathbb{R}^{n \times n}$ which is uniformly elliptic and $c_3 > 0$ (which is independent of $\lambda$) such that

$$
\int \sum_{kl} \left[ G^{kl}_\lambda(|\nabla u_\lambda|) \partial_t (\varphi_\lambda(|\nabla u_\lambda|)) \right] \partial_k \eta \, dx \leq -c \int \eta |\nabla V_\lambda(|\nabla u_\lambda|)|^2 dx \leq 0
$$

(5.7)
holds for all \( \eta \in C^1_0(2B), \eta \geq 0 \). Moreover,

\[
\min \{ c_0, 1 \} \| \xi \|^2 \leq \sum_{k,l} G^{kl}_\lambda(Q) \xi_k \xi_l \leq (c_1 + 1) \| \xi \|^2 \tag{5.8}
\]

for all \( Q \in \mathbb{R}^{n \times N} \) and all \( \xi \in \mathbb{R}^n \), where \( c_0, c_1 > 0 \) are the constants from (5.6), which depend on the constant in (2.7) but are independent of \( \lambda > 0 \).

Proof. Let \( \eta \in C^1_0(2B) \). Let \( B \) be a ball of radius \( R \) and let \( h \in \mathbb{R}^n \setminus \{0\} \) with \( |h| \leq \min \{ \text{dist}(\text{supp}(\eta), \partial(2B)), 1 \} \). Define \( \xi := |h|^{-2} \tau_h (\eta \tau_h u_\lambda) \), then \( \xi \in W^{1,\varphi}_0(2B) \), so \( \xi \) is an admissible test function for (3.1). This implies

\[
0 = \int |h|^{-2} \sum_{j,k} \tau_h (A^{jk}_\lambda(\nabla u_\lambda)) \partial_k (\eta \tau_h u_{\lambda,j}) \, dx
= \int |h|^{-2} \sum_{j,k} \tau_h (A^{jk}_\lambda(\nabla u_\lambda)) (\partial_k \eta) \tau_h u_{\lambda,j} \, dx
+ \int |h|^{-2} \sum_{j,k} \tau_h (A^{jk}_\lambda(\nabla u_\lambda)) \eta \tau_h \partial_k u_{\lambda,j} \, dx
=: (I) + (II).
\]

By Lemma 2.4 there exists \( c_3 > 0 \) (independent of \( \lambda \)) such that

\[
(II) \geq c_3 \int \eta \cdot |h|^{-2} |\tau_h V_\lambda(\nabla u_\lambda)|^2 \, dx =: c_3 (III). \tag{5.9}
\]

We choose \( h := re_l \) with \( l \in \{1, \ldots, n\} \) and \( 0 < r \leq \text{dist}(\text{supp}(\eta), \partial(2B)) \). Then with Corollary 3.7, we have for \( r \to 0 \)

\[
(III) \to \int \eta |\partial_l V_\lambda(\nabla u_\lambda)|^2 \, dx. \tag{5.10}
\]

We claim that for \( r \to 0 \)

\[
(I) \to \int \sum_{j,k} \partial_l (A^{jk}_\lambda(\nabla u_\lambda)) (\partial_k \eta) \partial_l u_{\lambda,j} \, dx \tag{5.11}
\]

and the integral is well defined in \( L^1 \). By Lemma 5.2, we have \( A_\lambda(\nabla u_\lambda) \in W^{1,s_2}(2B) \) for some \( s_2 > 1 \) and certainly we have \( u_\lambda \in W^{1,\varphi}(2B) \). So it follows (for a suitable subsequence) that

\[
(IV) := |h|^{-2} \sum_{j,k} \tau_h (A^{jk}_\lambda(\nabla u_\lambda)) (\partial_k \eta) \tau_h u_{\lambda,j}
\]

\[
\to \sum_{j,k} \partial_l (A^{jk}_\lambda(\nabla u_\lambda)) (\partial_k \eta) \partial_l u_{\lambda,j} =: (V) \tag{5.12}
\]

almost everywhere for \( r \to 0 \). So it remains to find a majorant for the left hand side of (5.12) that converges in \( L^1(2B) \) in order to conclude (5.11) by the dominated
convergence Theorem. For these calculations we can keep $\eta$ fixed, so the constants in the following estimates for $(IV)$ may depend on $\eta$. By Lemma 2.4 we have

$$|(IV)| \leq c |h|^{-2}|\tau_h(A_\lambda(\nabla u_\lambda))||\tau_h u_\lambda| \leq c |h|^{-2}(\varphi_\lambda)_{|\nabla u_\lambda|}(|\tau_h \nabla u_\lambda|) |\tau_h u_\lambda|.$$  

Now we proceed exactly as in [6] (see therein the estimate of $(I_2)$ in the proof of Lemma 12). Define $T_{\sigma e_l}: \mathbb{R}^n \to \mathbb{R}^n$ by $T_{\sigma e_l}(x) := x + \sigma e_l$. Then $\tau_h u_\lambda = \int_0^r \partial_t u_\lambda \circ T_{\sigma e_l} \, d\sigma$, so we estimate

$$|(IV)| \leq c |h|^{-2}(\varphi_\lambda)_{|\nabla u_\lambda|}(|\tau_h \nabla u_\lambda|) r \int_0^r |\nabla u_\lambda \circ T_{\sigma e_l}| \, d\sigma.$$  

With Young’s inequality (4.4) for $(\varphi_\lambda)_{|\nabla u_\lambda|}$ and Jensen’s inequality we get

$$|(IV)| \leq c |h|^{-2}(\varphi_\lambda)_{|\nabla u_\lambda|}(|\tau_h \nabla u_\lambda|) + c |h|^{-2} \int_0^r (\varphi_\lambda)_{|\nabla u_\lambda|}(r |\nabla u_\lambda \circ T_{\sigma e_l}|) \, d\sigma.$$  

By (4.6), (4.5), and Lemma 2.4 we have

$$(\varphi_\lambda)_{|\nabla u_\lambda|}(r |\nabla u_\lambda \circ T_{\sigma e_l}|) \leq c (\varphi_\lambda)_{|\nabla u_\lambda|} T_{\sigma e_l}(r |\nabla u_\lambda \circ T_{\sigma e_l}|) + c |V|(|\nabla u_\lambda| - V(\nabla u_\lambda) \circ T_{\sigma e_l})^2 \leq c r^2 (\varphi_\lambda)_{|\nabla u_\lambda|} T_{\sigma e_l} + c |\tau_{\sigma e_l} V(\nabla u_\lambda)|^2 \leq c r^2 |V(\nabla u_\lambda \circ T_{\sigma e_l})|^2 + c |\tau_{\sigma e_l} V(\nabla u_\lambda)|^2.$$  

So with the previous estimate and $|h| = r$ we get

$$|(IV)| \leq c |h|^{-2}|\tau_h \nabla V(\nabla u_\lambda)|^2 + c \int_0^{|h|} |V_\lambda(\nabla u_\lambda \circ T_{\sigma e_l})|^2 \, d\sigma$$  

$$+ c |h|^{-2} \int_0^{|h|} |\tau_{\sigma e_l} V(\nabla u_\lambda)|^2 \, d\sigma$$  

$$\leq c |h|^{-2}|\tau_h \nabla V(\nabla u_\lambda)|^2 + c \int_0^{|h|} |V_\lambda(\nabla u_\lambda \circ T_{\sigma e_l})|^2 \, d\sigma$$  

$$+ c \int_0^{|h|} |\sigma|^{-2}|\tau_{\sigma e_l} V(\nabla u_\lambda)|^2 \, d\sigma.$$  

From Corollary 3.7 it follows that $|h|^{-1} \tau_h V_\lambda(\nabla u_\lambda)$ converges to $\partial_t V_\lambda(\nabla u_\lambda)$ in $L^2(2B)$ for $|h| \to 0$. As a consequence $|h|^{-2}|\tau_h \nabla V(\nabla u_\lambda)|^2 \to |\partial_t V_\lambda(\nabla u_\lambda)|^2$ in $L^1(2B)$ as $|h| \to 0$. We will show now that the second and the third term on the right-hand side converge to $|V_\lambda(\nabla u_\lambda)|^2$ and $|\partial_t V_\lambda(\nabla u_\lambda)|^2$ in $L^1(2B)$ for $|h| \to 0$, respectively. First, observe that $V_\lambda(\nabla u_\lambda \circ T_{\sigma e_l}) = V_\lambda(\nabla u_\lambda \circ T_{\sigma e_l}) \to V_\lambda(\nabla u_\lambda)$
in \( L^2(2B) \) for \( \sigma \to 0 \). Thus, \(|\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda \circ T_{\pi\sigma})|^2 \to |\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)|^2 \) in \( L^1(2B) \) for \( \sigma \to 0 \). Second, we have already shown that \(|\sigma|^{-2}|\tau_\sigma\mathbf{V}(\nabla\mathbf{u}_\lambda)|^2 \to |\partial_\lambda\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)|^2 \) in \( L^1(2B) \) as \( \sigma \to 0 \). So the second and the third term on the right-hand side both look like \( \int_0^{|h|} a_\sigma \, d\sigma \) with some function \( a_\sigma \in L^1(2B) \) with \( a_\sigma \to a \) in \( L^1(2B) \).

We claim that \( \int_0^{|h|} a_\sigma \, d\sigma \to a \) in \( L^1(2B) \). Indeed, for arbitrary \( \varepsilon > 0 \) there exists \( \sigma_0 > 0 \) such that \( \|a_\sigma - a\|_{L^1(2B)} < \varepsilon \) for all \( \sigma \in (0, \sigma_0) \). Now, we estimate for \( |h| \leq \sigma_0 \).

\[
\left\| \int_0^{|h|} a_\sigma \, d\sigma - a \right\|_{L^1(2B)} = \left\| \int_0^{|h|} a_\sigma - a \, d\sigma \right\|_{L^1(2B)} \leq \int_0^{|h|} \|a_\sigma - a\|_{L^1(2B)} \, d\sigma \leq \varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, this proves \( \int_0^{|h|} a_\sigma \, d\sigma \to a \) in \( L^1(2B) \) for \( |h| \to 0 \).

As a consequence we have found a majorant of \(\text{term to } |\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)|^2 \) and \( |\partial_\lambda\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda)|^2 \) in \( L^1(2B) \) for \( |h| \to 0 \), respectively. In total we have found a majorant of \((IV)\) that converges in \( L^1(2B) \). This and the already mentioned almost everywhere convergence prove our claim (5.11).

Using \((I) = -(II),(5.9),(5.10),\) and (5.11) we get after summation over \( l = 1, \ldots, n \)

\[
\int \sum_{l,j,k} \partial_l (A^{jk}_\lambda (\nabla\mathbf{u}_\lambda)) (\partial_k \eta) \partial_l \mathbf{u}_{\lambda,j} \, dx \leq -c_3 \int \eta |\nabla(\mathbf{V}_\lambda(\nabla\mathbf{u}_\lambda))|^2 \, dx \leq 0.
\]

(5.13)

Now, with Lemmas 4.3, 5.2 and 5.3, we calculate almost everywhere

\[
\sum_{j,l} \left( \partial_l (A^{jk}_\lambda (\nabla\mathbf{u}_\lambda)) \partial_l \mathbf{u}_{\lambda,j} \right)
= \sum_{j,l} \left[ \left( \frac{\varphi''(\lambda\mathbf{u}_\lambda)}{|\mathbf{u}_\lambda|^2} - \frac{\varphi'_\lambda(|\mathbf{u}_\lambda|)}{|\mathbf{u}_\lambda|^2} \right) \partial_l |\mathbf{u}_\lambda| \partial_k \mathbf{u}_{\lambda,j} \partial_l \mathbf{u}_{\lambda,j} + \varphi'_\lambda(|\mathbf{u}_\lambda|) \partial_l \partial_k \mathbf{u}_{\lambda,j} \partial_l \mathbf{u}_{\lambda,j} \right]
\]

\[
= \sum_{j,l} \left( \frac{\partial_k \mathbf{u}_{\lambda,j} \partial_l \mathbf{u}_{\lambda,j}}{|\mathbf{u}_\lambda|^2} \left( \varphi''(\lambda\mathbf{u}_\lambda) |\mathbf{u}_\lambda| - \varphi'_\lambda(|\mathbf{u}_\lambda|) \partial_l |\mathbf{u}_\lambda| \right) + \varphi'_\lambda(|\mathbf{u}_\lambda|) \partial_k |\mathbf{u}_\lambda| \right)
\]

\[
= \sum_{j,l} \left( \frac{\partial_k \mathbf{u}_{\lambda,j} \partial_l \mathbf{u}_{\lambda,j}}{|\mathbf{u}_\lambda|^2} \omega_\lambda(|\mathbf{u}_\lambda|) \varphi'_\lambda(|\mathbf{u}_\lambda|) \partial_l |\mathbf{u}_\lambda| \right) + \varphi'_\lambda(|\mathbf{u}_\lambda|) \partial_k |\mathbf{u}_\lambda| \right)
\]

\[
= \sum_{j,l} \left( \frac{\partial_k \mathbf{u}_{\lambda,j} \partial_l \mathbf{u}_{\lambda,j}}{|\mathbf{u}_\lambda|^2} \omega_\lambda(|\mathbf{u}_\lambda|) \partial_l (\varphi'_\lambda(|\mathbf{u}_\lambda|)) \right) + \partial_k (\varphi'_\lambda(|\mathbf{u}_\lambda|)),
\]

where \( \omega_\lambda : [0, \infty) \to \mathbb{R} \) is given by (5.5). Define \( G^{kl}_\lambda : \mathbb{R}^{N \times n} \to \mathbb{R}^{n \times n} \) by

\[
G^{kl}_\lambda (\mathbf{Q}) := \delta_{k,l} + \sum_{j} \frac{(Q_{jk} Q_{jl})}{|\mathbf{Q}|^2} \omega_\lambda(|\mathbf{Q}|).
\]
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Then

$$\sum_{jl} \left( \partial_l (A_{jk}^l (\nabla u) \partial_l u_{\lambda, j}) \right) = \sum_l G_{\lambda}^{kl} (\nabla u_{\lambda}) \partial_l (\varphi_{\lambda}(|\nabla u_{\lambda}|)).$$

This together with (5.13) implies

$$\int \sum_{kl} \left[ G_{\lambda}^{kl} (\nabla u_{\lambda}) \partial_l (\varphi_{\lambda}(|\nabla u_{\lambda}|)) \right] \partial_k \eta \, dx \leq -c \int \eta |\nabla V_{\lambda}(\nabla u_{\lambda})|^2 \, dx \leq 0.$$

This proves (5.7). For all $Q \in \mathbb{R}^{N \times n}$ and all $\xi \in \mathbb{R}^n$ holds

$$\sum_{k,l} G_{\lambda}^{kl} (Q) \xi_k \xi_l = |\xi|^2 + \frac{|Q|}{|Q|^2} \omega_{\lambda}(|Q|).$$

This implies

$$\sum_{k,l} G_{\lambda}^{kl} (Q) \xi_k \xi_l \leq |\xi|^2 (c_0 + 1) |\xi|^2,$$

$$\sum_{k,l} G_{\lambda}^{kl} (Q) \xi_k \xi_l \geq |\xi|^2 (1 + \min \{0, c_0 - 1\}) = \min \{c_0, 1\},$$

where $c_0$ and $c_1$ are the constants from (5.6).

Let us transfer this result to our original system by passing to the limit $\lambda \to 0$.

**Theorem 5.5.** Let $\varphi$ satisfy Assumption 2.1, let $u$ be a local minimizer of the functional (1.1), and let $B$ be a ball with $4B \subset \Omega$. Then there exists $G : 2B \to \mathbb{R}^{n \times n}$ which is uniformly elliptic, i.e.

$$\min \{c_0, 1\} |\xi|^2 \leq \sum_{k,l} G^{kl} (Q) \xi_k \xi_l \leq (c_1 + 1) |\xi|^2,$$

for all $Q \in \mathbb{R}^{n \times n}$ and all $\xi \in \mathbb{R}^n$, where $c_0, c_1 > 0$ only depend on the constant in (2.7), such that

$$\int \sum_{kl} \left[ G^{kl} (\nabla u) \partial_l (\varphi(|\nabla u|)) \right] \partial_k \eta \, dx \leq -c \int \eta |\nabla V(\nabla u)|^2 \, dx \leq 0$$

(5.14)

holds for all $\eta \in C^1_0 (2B), \eta \geq 0$.

**Proof.** We proceed as in the proof of Theorem 4.9. In particular, let $u_{\lambda}$ be the minimizer of $F_{\lambda}(w) = \int_{4B} \varphi_{\lambda}(|\nabla w|) \, dx$ within the class $\{w \in W^{1,\varphi}(4B) : w = u$ on $\partial(4B)\}$. Again we restrict ourselves to the case $n \geq 3$. The other case follows analogously. From the proof of Theorem 4.9 we know that $V_{\lambda}(\nabla u_{\lambda}) \to V(\nabla u)$ in $W^{1,2}(2B)$ and $\nabla \varphi_{\lambda}(|\nabla u_{\lambda}|) \to \nabla \varphi(|\nabla u|)$ in $L^{n+1}_{\text{loc}} (2B)$.

For any $\lambda > 0$ there exists by Lemma 5.4 a function $G_{\lambda} : 2B \to \mathbb{R}^{n \times n}$ such that (5.7) holds for all $\eta \in C^1_0 (4B)$. These $G_{\lambda}$ satisfy (5.8) and are therefore uniformly elliptic independent of the choice of $\lambda$. In particular, there exists a sequence
\( \lambda_j \to 0 \) such that \( G_{\lambda_j} \) converges almost everywhere to some \( G : 2B \to \mathbb{R}^{n \times n} \) which is also uniformly elliptic and satisfies (5.8). So for \( \eta \in C^1_0(2B) \) the term \( G^{kl}_{\lambda_j}(\nabla u_\lambda) \partial_k \eta \) in (5.7) converges strongly in \( L^n(2B) \) to \( G^{kl}(\partial_k \eta) \) for \( \lambda \to \infty \). This and \( \nabla \varphi_\lambda(\nabla u_\lambda) \to \nabla \varphi(|\nabla u|) \) in \( L^{n-1}(2B) \) proves the convergence of the first integral in (5.7) to the first integral in (5.14) for \( j \to \infty \). The convergence of the second integral for \( j \to \infty \) follows by the lower semicontinuity of the integral with respect to weak convergence in \( L^2(2B) \) and \( \nabla V_\lambda(\nabla u_\lambda) \to \nabla V(\nabla u) \) in \( L^2(2B) \).

It is obvious that the limit is non-positive. \( \square \)

Due to Theorem 5.5 we know that \( \varphi(|\nabla u|) \) is a subsolution of a uniformly elliptic equation. If \( \varphi(|\nabla u|) \in W^{1,\infty}_{\text{loc}}(\Omega) \), then we could apply Harnack’s inequality to get local \( L^\infty \) estimates. Unfortunately, we only know so far that \( \varphi(|\nabla u|) \in W^{1,\frac{n}{n-1}}_{\text{loc}}(\Omega) \), see Theorem 4.9. To overcome this difficulty, we need the following result due to Marcellini and Papi, proved under more general hypotheses.

**Proposition 5.6.** [22, Theorem A] Let \( \varphi \) satisfy Assumption 2.1. Let \( \lambda > 0 \) and let \( u_\lambda \) be a local minimizer of the functional (4.1). Then \( u_\lambda \in W^{1,\infty}_{\text{loc}}(\Omega) \).

Note that we need Proposition 5.6 only to justify \( \varphi_\lambda(|\nabla u_\lambda|) \in W^{1,2}_{\text{loc}}(\Omega) \). We do not use any qualitative estimates for \( u_\lambda \) in \( W^{1,\infty}_{\text{loc}}(\Omega) \).

**Lemma 5.7.** Let \( \varphi \) satisfy Assumption 2.1. Let \( \lambda > 0 \) and let \( u_\lambda \) be a local minimizer of the functional (4.1). Then \( u_\lambda \in W^{2,2}_{\text{loc}}(\Omega) \) and \( \varphi_\lambda(|\nabla u_\lambda|) \in W^{1,2}_{\text{loc}}(\Omega) \).

**Proof.** Let \( B \) be a ball with \( 2B \subseteq \Omega \). Due to Proposition 5.6 we have \( u_\lambda \in W^{1,\infty}(B) \). Let \( M := \|\nabla u_\lambda\|_{L^{\infty}(B)} \). Due to Lemma 4.3 and Corollary 4.6 we have \( u_\lambda \in W^{2,s_1}_{\text{loc}}(\Omega) \) and \( \varphi_\lambda(|\nabla u_\lambda|) \in W^{1,s_3}_{\text{loc}}(\Omega) \) for some \( s_1, s_3 > 1 \). So by Corollary 3.7 we have \( V_\lambda(\nabla u_\lambda) \in W^{1,2}(B) \). So with Lemma 4.4 and \( \varphi''(\lambda + t) \sim \varphi''(\lambda + t) \sim \varphi(\lambda + t)/(\lambda + t)^2 \) uniformly in \( \lambda > 0 \) and \( t \geq 0 \) we deduce that

\[
\infty > \int_B \varphi''(|\nabla u_\lambda|)|\nabla^2 u_\lambda|^2 \chi(|\nabla u_\lambda| \leq M) \, dx \geq c \frac{\varphi(\lambda)}{(\lambda + M)^2} \int_B |\nabla^2 u_\lambda|^2 \, dx.
\]

This proves \( \nabla u_\lambda \in W^{2,2}(B) \). With Lemma 4.5 we estimate

\[
\int_B |\nabla \varphi_\lambda(|\nabla u_\lambda|)|^2 \, dx \leq c \int_B (\varphi'_\lambda(|\nabla u_\lambda|))^2 |\nabla^2 u_\lambda|^2 \, dx
\]

\[
\leq c (\varphi'_\lambda(M))^2 \int_B |\nabla^2 u_\lambda|^2 \, dx < \infty.
\]

This proves \( \varphi_\lambda(|\nabla u_\lambda|) \in W^{1,2}(B) \). \( \square \)

Using the previous lemma we can now provide the desired \( W^{1,2} \)-estimate for \( \varphi(|\nabla u|) \).
Lemma 5.8. Let \( \varphi \) satisfy Assumption 2.1, let \( u \) be a local minimizer of the functional (1.1), and let \( B \) be a ball with \( 2B \Subset \Omega \). Then

\[
\sup_B \varphi(|\nabla u|) \leq c \int_{2B} \varphi(|\nabla u|) \, dx, \tag{5.15}
\]

\[
\int_B |\nabla \varphi(|\nabla u|)|^2 \, dx \leq c R^{-2} \int_{2B} |\varphi(|\nabla u|)|^2 \, dx. \tag{5.16}
\]

Proof. First, we prove the lemma with \( \varphi \) replaced by \( \varphi_\lambda \). Due to Lemma 5.7 we can apply Harnack’s inequality to (5.7) to get (5.15). The estimate (5.16) follows as usual with the test function \( \eta := \kappa \varphi_\lambda(|\nabla u_\lambda|) \) in Lemma 5.4, where \( \kappa \in C^\infty_0(2B) \) with \( \chi_B \leq \kappa \leq \chi_{2B} \) and \( \|\nabla \kappa\|_\infty \leq c/R \). Now, exactly as in the proof of Theorems 4.9 and 5.5 the claim follows by passing to the limit \( \lambda \to 0 \). \( \square \)

We will now apply the theory of subsolutions to \( \varphi(|\nabla u|) \) to derive the weak Harnack inequality.

Theorem 5.9. Let \( \varphi \) satisfy Assumption 2.1. Let \( u \) be a local minimizer of the functional (1.1), and let \( B \) be a ball such that \( 2B \Subset \Omega \). Then

\[
\Phi(u, \frac{1}{2}B) \leq c \left( \sup_B \varphi(|\nabla u|) - \sup_{\frac{1}{2}B} \varphi(|\nabla u|) \right) \tag{5.17}
\]

where \( \Phi(u, B) := \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 \, dx \) is the excess functional.

Proof. Due to Theorem 5.5 we know that \( \varphi(|\nabla u|) \) is a subsolution of the uniformly elliptic equation (5.14). Moreover, by Lemma 5.8 we know that \( \varphi(|\nabla u|) \in W^{1,2}_{\text{loc}}(\Omega) \). Therefore, we can proceed exactly as in [1] and [15, Proposition 3.1] and apply Harnack’s inequality for subsolutions to get (5.17). \( \square \)

6. Hölder continuity of the gradients

In this section we will prove that \( V(\nabla u) \) and \( \nabla u \) are Hölder continuous. For this we have to strengthen our requirements on \( \varphi \). In particular, we assume that \( \varphi \) satisfies Assumption 2.2, which states that \( \varphi'' \) is Hölder continuous away from zero. We will use this property of \( \varphi \) in form of additional regularity of \( A \), which is expressed in the following lemma.

Lemma 6.1. Let \( \varphi \) satisfy Assumption 2.2 and let \( \beta > 0 \) be as Assumption 2.2. Then there exists \( c > 0 \) such that for all \( H, Q \in \mathbb{R}^{N \times n} \) with \( |H| < \frac{1}{2}|Q| \) holds

\[
|\nabla_{N \times n} A(Q + H) - \nabla_{N \times n} A(Q)| \, ds \leq c \varphi''(|Q|) \left( \frac{|H|}{|Q|} \right)^\beta,
\]

where \( c \) depends on \( \varphi \) only via the constants in (2.7) and (2.8).

Proof. The claim follows immediately from (2.19), (2.8), and Remark 2.3. \( \square \)
Following the ideas of Lemma 2.10 in [1] we prove the following excess decay estimate.

**Lemma 6.2.** Let \( \varphi \) satisfy Assumption 2.2 and let \( u \) be a local minimizer of the functional (1.1). Then there exists \( c > 1 \) such that for every \( \tau \in (0, 1) \) there exists \( \varepsilon_0 > 0 \) such that for every ball \( B \Subset \Omega \) holds

\[
\Phi(u, B) \leq \varepsilon_0 \sup_{\frac{1}{4}B} \varphi(|\nabla u|) \implies \Phi(u, \tau B) \leq c \tau^2 \Phi(u, B).
\]

Note that \( c \) and \( \varepsilon_0 \) depend on \( \varphi \) only via the constant in (2.7).

**Proof.** We fix \( \tau \in (0, 1) \) and will choose \( \varepsilon_0 > 0 \) later. Without loss of generality we can assume \( \tau \leq \frac{1}{2} \). Let \( B \Subset \Omega \) be a ball such that \( \Phi(u, B) \leq \varepsilon_0 \sup_{\frac{1}{2}B} \varphi(|\nabla u|) \).

Let \( Q \in \mathbb{R}^{N \times n} \) be such that

\[
\mathbf{V}(Q) = \langle \nabla \mathbf{u} \rangle_B
\]

From the Lemma 5.8, Lemma 2.4, and the definition of \( \Phi(u, B) \) we get

\[
\sup_{\frac{1}{2}B} \varphi(|\nabla u|) \leq c \int_B \varphi(|\nabla u|) \, dx \leq c \int_B |\mathbf{V}(\nabla \mathbf{u})|^2 \, dx \leq c (\Phi(u, B) + |\mathbf{V}(Q)|^2).
\]

So if \( \varepsilon_0 < \frac{1}{2}c \) we can conclude

\[
\sup_{\frac{1}{2}B} \varphi(|\nabla u|) \leq c |\mathbf{V}(Q)|^2 \leq c \varphi(|Q|), \tag{6.1}
\]

where we have also used Lemma 2.4.

If \( Q = 0 \), then we get by (6.1) that \( \nabla u = 0 \) on \( \frac{1}{2}B \) and the conclusion of the lemma is immediate. So in the following we can assume that \( Q \neq 0 \).

Since \( \varphi \) is strictly increasing, we deduce from (6.1) that

\[
\sup_{\frac{1}{2}B} |\nabla u| \leq c |Q|, \tag{6.2}
\]

where we have also used that \( \Delta_2(\varphi) < \infty \). This proves \( |\nabla u| + |Q| \sim |Q| \) on \( \frac{1}{2}B \), so with Lemma 2.4, \( \varphi(t) \sim \varphi''(t) t^2 \), and \( \Delta_2(\varphi) < \infty \), we get

\[
\Phi(u, \frac{1}{2}B) \sim \int_{\frac{1}{2}B} |\mathbf{V}(\nabla \mathbf{u}) - \mathbf{V}(Q)|^2 \, dx
\]

\[
\sim \int_{\frac{1}{2}B} \varphi''(|Q| + |\nabla u|)|\nabla u - Q|^2 \, dx
\]

\[
\sim \int_{\frac{1}{2}B} \varphi''(|Q|)|\nabla u - Q|^2 \, dx. \tag{6.3}
\]
Everywhere regularity of functionals with \( \varphi \)-growth

We define \( w := u - q \), where \( q : \Omega \to \mathbb{R}^n \) is a linear function such that \( \nabla q = Q \) and \( \int_{\tau B} u - q \, dx = 0 \). Note that (6.2) implies that

\[
|\nabla w| = |\nabla u - Q| \leq c |Q| \quad \text{on } \frac{1}{2} B. \tag{6.4}
\]

Let \( v \) be the unique solution in \( w + W^{1,2}_0(\frac{1}{4} B, \mathbb{R}^N) \) satisfying

\[
\sum_{ij\alpha\gamma} \int_{\frac{1}{4} B} \partial_{j\gamma} A_{i\alpha}(Q) \partial_{\gamma} v_j \partial_{\alpha} z_i \, dx = 0 \tag{6.5}
\]

for all \( z \in W^{1,2}_0(\frac{1}{4} B, \mathbb{R}^N) \). Then by Remark 2.12 the system (6.5) is uniformly elliptic with lower and upper constants of ellipticity proportional to \( \varphi''(|Q|) \). Thus by the theory of elliptic systems we have (with constants independent of \( Q \))

\[
\int_{\tau B} |\nabla v - \langle \nabla v \rangle_{\tau B}|^2 \, dx \leq c \tau^2 \int_{\frac{1}{4} B} |\nabla v - \langle \nabla v \rangle_{\frac{1}{4} B}|^2 \, dx \tag{6.6}
\]

for all \( \tau \in (0, \frac{1}{4}] \). From (3.2), \( \nabla u = w + Q \), and Taylor’s formula we get

\[
0 = \sum_{i\alpha} \int_{\frac{1}{4} B} (A_{i\alpha}(\nabla u) - A_{i\alpha}(Q)) \partial_{\alpha} z_i \, dx
\]

\[
= \sum_{ij\alpha\gamma} \int_{\frac{1}{4} B} \int_0^1 \partial_{j\gamma} A_{i\alpha}(Q + s \nabla w(x)) \, ds \partial_{\gamma} w_j \partial_{\alpha} z_i \, dx
\]

for all \( z \in W^{1,2}_0(\frac{1}{4} B) \). In combination with (6.5) this implies

\[
\sum_{ij\alpha\gamma} \int_{\frac{1}{4} B} \partial_{j\gamma} A_{i\alpha}(Q)(\partial_{\gamma} v_j - \partial_{\gamma} w_j) \partial_{\alpha} z_i \, dx
\]

\[
= \sum_{ij\alpha\gamma} \int_{\frac{1}{4} B} \int_0^1 \left[ \partial_{j\gamma} A_{i\alpha}(Q + s \nabla w(x)) - \partial_{j\gamma} A_{i\alpha}(Q) \right] ds \partial_{\gamma} w_j \partial_{\alpha} z_i \, dx. \tag{6.7}
\]

Our goal is to estimate the right-hand side of (6.7). For fixed \( x \in \frac{1}{4} B \)

\[
(I) := \sum_{ij\alpha\gamma} \int_0^1 \left| \partial_{j\gamma} A_{i\alpha}(Q + s \nabla w(x)) - \partial_{j\gamma} A_{i\alpha}(Q) \right| ds.
\]
Let $\beta > 0$ be as in Assumption 2.2 and $q_3 > 1$ as in Lemma 3.5. Define $\beta_2 := \min \{\beta, q_3 - 1\}$. We claim that

$$(I) \leq c \varphi''(|Q|) \left(\frac{|\nabla w(x)|}{|Q|}\right)^{\beta_2}. \quad (6.8)$$

**Case** $|\nabla w(x)| < \frac{1}{2}|Q|$: Then $s|\nabla w(x)| < \frac{1}{2}|Q|$ for all $s \in [0, 1]$. So with Lemma 6.1 we estimate

$$(I) \leq c \int_0^1 \varphi''(|Q|) \left(\frac{|\nabla w(x)|}{|Q|}\right)^{\beta_2} ds \leq c \varphi''(|Q|) \left(\frac{|\nabla w(x)|}{|Q|}\right)^{\beta_2}.$$

**Case** $|\nabla w(x)| \geq \frac{1}{2}|Q|$: Then (6.4) implies $|\nabla w(x)| \sim |Q|$. We estimate

$$(I) \leq \sum_{i,j,a,b} \int_0^1 |\partial_{j\gamma} A_{ia}(Q + s\nabla w(x))| + |\partial_{j\gamma} A_{ia}(Q)| ds.$$

So with (2.19), Lemma 2.7, (2.7), $\Delta_2(\varphi) < \infty$, and $|\nabla w(x)| \sim |Q|$ we get

$$(I) \leq c \left(\varphi''(|Q| + |\nabla w|) + \varphi''(|Q|)\right) \leq \varphi''(|Q|) \leq c \varphi''(|Q|) \left(\frac{|\nabla w|}{|Q|}\right)^{\beta}.$$

So we have proved our claim (6.8) in both cases.

We choose $z = v - w$ in (6.7) and with (6.8) we estimate

$$\int_{\frac{1}{2}B} \varphi''(|Q|) |\nabla v - \nabla w|^2 dx \leq c \int_{\frac{1}{2}B} \varphi''(|Q|) |Q|^{-\beta_2} |\nabla w|^{1+\beta_2} |\nabla v - \nabla w| dx,$$

where we have used again that (6.5) is uniformly elliptic with lower and upper constants of ellipticity proportional to $\varphi''(|Q|)$. It follows that

$$\int_{\frac{1}{2}B} \varphi''(|Q|) |\nabla v - \nabla w|^2 dx \leq c \varphi''(|Q|) |Q|^{-2\beta_2} \int_{\frac{1}{2}B} |\nabla w|^{2+2\beta_2} dx.$$

Note that due to (6.2), $\nabla w = \nabla u - Q$ on $\frac{1}{2}B$, and Lemma 2.4 we have

$$\varphi''(|Q|)|\nabla w|^2 \sim \varphi''(|Q| + |\nabla u||)|\nabla u - Q|^2 \sim |V(\nabla u) - V(Q)|^2. \quad (6.9)$$

So with the previous estimate and $\varphi(t) \sim \varphi''(t)t^2$ (see (2.4) and (2.7)) we get

$$\int_{\frac{1}{2}B} \varphi''(|Q|) |\nabla v - \nabla w|^2 dx \leq c \varphi(|Q|)^{-\beta_2} \int_{\frac{1}{2}B} |V(\nabla u) - V(Q)|^{2(1+\beta_2)} dx.$$
Now with Jensen’s inequality and Lemma 3.5 (using $1 \leq 1 + \beta_2 \leq q_3$) we deduce
\[
\int_{\frac{1}{2}B} \varphi''(|Q|)|\nabla v - \nabla w|^2 \, dx \leq c \left( \varphi(|Q|) \right)^{-\beta_2} \int_{\frac{1}{2}B} |V(\nabla u) - V(Q)|^{2(1+\beta_2)} \, dx \\
\leq c \left( \varphi(|Q|) \right)^{-\beta_2} \left( \int_{\frac{1}{2}B} |V(\nabla u) - V(Q)|^2 \, dx \right)^{1+\beta_2}.
\]
(6.10)

We will use this estimate later in our calculations. Let us now start to estimate $\Phi(u, \tau B)$. Using Lemma 2.4 we estimate
\[
\Phi(u, \tau B) = \int_{\tau B} |V(\nabla u) - \langle V(\nabla u) \rangle_{\tau B}|^2 \, dx \\
= \inf_{H \in \mathbb{R}^{B \times N}} \int_{\tau B} |V(\nabla u) - H|^2 \, dx \\
\leq \int_{\tau B} |V(\nabla u) - V(\langle \nabla u \rangle_{\tau B})|^2 \, dx \\
\leq c \int_{\tau B} \varphi''(|\nabla u| + |\langle \nabla u \rangle_{\tau B}|) |\nabla u - \langle \nabla u \rangle_{\tau B}|^2 \, dx \\
= c \int_{\tau B} \varphi''(|\nabla u| + |\langle \nabla u \rangle_{\tau B}|) |\nabla w - \langle \nabla w \rangle_{\tau B}|^2 \, dx. \quad (6.11)
\]

We claim that
\[
|\nabla u| + |\langle \nabla u \rangle_{\tau B}| \sim |Q|. \quad (6.12)
\]

From (6.2) it is clear that $|\nabla u| + |\langle \nabla u \rangle_{\tau B}| \leq c |Q|$ on $\tau B$ (using $\tau \leq \frac{1}{2}$). To show the converse, we will show that $|\langle \nabla u \rangle_{\tau B}| \geq c |Q|.$

\[
\varphi(|Q|) \leq c \varphi''(|Q|)|Q|^2 \\
\leq c \varphi''(|Q|) \left( |\langle \nabla u \rangle_{\tau R} - Q|^2 + |\langle \nabla u \rangle_{\tau R}|^2 \right) \\
\leq c \varphi''(|Q|) \left( \int_{\tau B} |\nabla u - Q|^2 \, dx + |\langle \nabla u \rangle_{\tau B}|^2 \right) \\
\leq c \varphi''(|Q|) \left( \tau^{-n} \int_{B} |\nabla u - Q|^2 \, dx + |\langle \nabla u \rangle_{\tau B}|^2 \right).
\]

Now, (6.3), our assumption $\Phi(u, B) \leq \varepsilon_0 \sup_{\frac{1}{2}B} \varphi(|\nabla u|)$, and (6.1) imply

\[
\varphi(|Q|) \leq c \left( \tau^{-n} \varepsilon_0 \sup_{\frac{1}{2}B} \varphi(|\nabla u|) + \varphi''(|Q|)|\langle \nabla u \rangle_{\tau B}|^2 \right) \\
\leq c \left( \tau^{-n} \varepsilon_0 \varphi(|Q|) + \varphi''(|Q|)|\langle \nabla u \rangle_{\tau B}|^2 \right).
\]
Now, this is the first of two places where we need to choose $\varepsilon_0 > 0$ small (depending on $\tau$). If $\tau^{-n}\varepsilon_0 < \frac{1}{2\varepsilon}$, then we obtain (using $\varphi''(|Q|) |Q|^2 \sim \varphi(|Q|))$ that

$$|Q|^2 \leq c \left(\langle \nabla u \rangle_{\tau R}\right)^2.$$  

This was the missing step to prove (6.12). Now, combining (6.12) with (6.11) (using also $\varphi''(t) t^2 \sim \varphi(t)$ and $\Delta_2(\varphi) < \infty$) we get

$$\Phi(u, \tau B) \leq c \int_{\tau B} \varphi''(|Q|) |\nabla w - \langle \nabla w \rangle_{\tau B}|^2 \, dx. \quad (6.13)$$

We estimate

$$\int_{\tau B} |\nabla w - \langle \nabla w \rangle_{\tau B}|^2 \, dx$$

$$\leq 2 \int_{\tau B} \left[ |\nabla v - \langle \nabla v \rangle_{\tau B}|^2 + |\nabla v - \nabla w|^2 \right] \, dx$$

$$\leq c \left( \tau^2 \int_{\frac{1}{4} B} |\nabla v - \langle \nabla v \rangle_{\frac{1}{4} B}|^2 \, dx + \tau^{-n} \int_{\frac{1}{4} B} |\nabla v - \nabla w|^2 \, dx \right)$$

$$\leq c \left( \tau^2 \int_{\frac{1}{4} B} |\nabla w - \langle \nabla w \rangle_{\frac{1}{4} B}|^2 \, dx + \tau^{-n} \int_{\frac{1}{4} B} |\nabla v - \nabla w|^2 \, dx \right).$$

Now, combining (6.13), the previous estimate, $\nabla w = \nabla u - Q$, (6.10), and (6.3) we get

$$\Phi(u, \tau B) \leq c \tau^2 \varphi''(|Q|) \int_{\frac{1}{2} B} |\nabla u - Q|^2 \, dx + c \tau^{-n} c \left( \varphi(|Q|) \right)^{-\beta_2}$$

$$\times \left( \int_{\frac{1}{2} B} |V(\nabla u) - V(Q)|^2 \, dx \right)^{1+\beta_2}$$

$$\leq c \tau^2 \Phi(u, B) + c \tau^{-n} \left( \varphi(|Q|) \right)^{-\beta_2} \left( \Phi(u, B) \right)^{1+\beta_2}.$$  

Using $\Phi(u, B) \leq \varepsilon_0 \sup_{\frac{1}{2} B} \varphi(|\nabla u|)$ and (6.1) we estimate

$$\Phi(u, \tau B) \leq c \left( \tau^2 + \tau^{-n} \varepsilon_0^{\beta_2} \right) \Phi(u, B).$$

Now, this is the second place where we need to choose $\varepsilon_0 > 0$ small (depending on $\tau$). If we additionally assume $\varepsilon_0 \leq \tau^{\frac{2+n}{\beta_2}}$, then

$$\Phi(u, \tau B) \leq c_2 \tau^2 \Phi(u, B) \quad (6.14)$$

for some $c_2 > 0$. Note that $c_2$ does not depend on $\tau$.  \(\Box\)
Exactly as in [15, Proposition 3.2], we can remove the hypotheses in Lemma 6.2 by proving the following lemma.

**Lemma 6.3.** Let \( \varphi \) satisfy Assumption 2.2 and let \( u \) be a local minimizer of the functional (1.1). Then there exists \( c > 1 \) such that for every \( \tau \in (0, 1) \) there exists \( \varepsilon_0 > 0 \) and \( \delta \in (0, 1) \) such that for every ball \( B \subset \Omega \) one of the following alternatives holds:

(a) \( \Phi(u, \tau B) \leq c \tau^2 \Phi(u, B) \),
(b) \( \Phi(u, B) > \varepsilon_0 \sup_{\frac{1}{2} B} \varphi(|\nabla u|) \) and \( \sup_{\frac{1}{4} B} \varphi(|\nabla u|) \leq \delta \sup_{\frac{1}{2} B} \varphi(|\nabla u|) \).

**Proof.** Fix \( \tau \in (0, 1) \). Without loss of generality we can assume \( \tau < \frac{1}{4} \). Choose \( \varepsilon_0 > 0 \) as in Lemma 6.2. We will choose \( \delta > 0 \) later.

If the second alternative (b) is not true, then

\[
\Phi(u, B) \leq \varepsilon_0 \sup_{\frac{1}{2} B} \varphi(|\nabla u|) \tag{6.15}
\]

or

\[
\Phi(u, B) > \varepsilon_0 \sup_{\frac{1}{2} B} \varphi(|\nabla u|) \quad \text{and} \quad \sup_{\frac{1}{4} B} \varphi(|\nabla u|) > \delta \sup_{\frac{1}{2} B} \varphi(|\nabla u|) \tag{6.16}
\]

In the case of (6.15) we deduce from Lemma 6.2 that (a) holds. So in the following assume that (6.16) holds. The Harnack inequality (5.17) then yields

\[
\Phi\left(u, \frac{1}{4} B\right) \leq c \left( \sup_{\frac{1}{2} B} \varphi(|\nabla u|) - \sup_{\frac{1}{4} B} \varphi(|\nabla u|) \right)
\leq c \left( 1 - \delta \right) \sup_{\frac{1}{2} B} \varphi(|\nabla u|) \leq c \frac{1 - \delta}{\varepsilon_0} \Phi(u, B)
\]

and therefore

\[
\Phi(u, \tau B) \leq c \tau^{-n} \Phi\left(u, \frac{1}{4} B\right) \leq c \tau^{-n} \frac{1 - \delta}{\varepsilon_0} \Phi(u, B)
\]

So choosing \( \delta \) close to 1 in such a way that \( c \tau^{-n} \frac{1 - \delta}{\varepsilon_0} < \tau^2 \), we conclude again that

\[
\Phi(u, \tau B) \leq \tau^2 \Phi(u, B)
\]

We notice that \( \delta \) depends on \( \tau \), but not on \( B \). \( \square \)

Now, based on Lemma 6.3 the standard iteration technique (see the proof of Theorem 3.1 in [15]) allows us to conclude with the following theorem.

**Theorem 6.4.** Let \( \varphi \) satisfy Assumption 2.2 and let \( u \) be a local minimizer of the functional (1.1). Then there exists \( \sigma > 0 \) such that for all balls \( B \subset \Omega \) and all \( \lambda \in (0, 1) \) holds

\[
\Phi(u, \lambda B) \leq c \lambda^\sigma \Phi(u, B).
\]
From Campanato’s characterization of Hölder continuous functions, see chapter III of [14], we immediately conclude the local $\sigma$-Hölder continuity of $V(\nabla u)$. So together with Lemma 2.10 we get the following result.

**Theorem 6.5.** Let $\varphi$ satisfy Assumption 2.2 and let $u$ be a local minimizer of the functional (1.1). Then there exists $\sigma > 0$ such that $V(\nabla u)$, $\nabla u$, and $A(\nabla u)$ are locally $\sigma$-Hölder continuous.

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**References**


