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Fractional Estimates for Non-Differentiable Elliptic Systems with general Growth

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Abstract In this paper we study the regularity of weak solutions of the elliptic system $-\text{div}(A(x,\nabla u)) = b(x,\nabla u)$ with non-standard $\varphi$-growth condition. Here $\varphi$ is a given Orlicz function. We are interested in the case where $A$ and $b$ are not differentiable with respect to $x$ but only H"older continuous with exponent $\alpha$. We show that the natural quantity $V(\nabla u)$ is locally in the Nikolski\t space $N^{\alpha,2}$. From this it follows that the set of singularities of $V(\nabla u)$ has Hausdorff dimension less or equal $n - 2\alpha$, where $n$ is the dimension of the domain $\Omega$. One of the main features of our technique is that it handles the case of the $p$-Laplacian for $1 < p < \infty$ in a unified way. There is no need to use different approaches for the cases $p \leq 2$ and $p \geq 2$.

Keywords Elliptic Systems; Singular set; Hausdorff dimension; Orlicz Function; non-differentiable

Mathematics Subject Classification (2000) 35J60; 35D10

1 Introduction

In this paper we are concerned with fractional estimates for weak solutions of the system

$$-\text{div}(A(x,\nabla u)) = b(x,\nabla u) \quad \text{in} \, \Omega \tag{1.1}$$
where $\Omega \subset \mathbb{R}^n$ is a bounded, open domain. We assume that the elliptic operator satisfies non-standard $\varphi$-growth and $\varphi$-monotonicity conditions, i.e.

$$\begin{align*}
(A(x,P) - A(x,Q)) \cdot (P - Q) &\geq c \varphi''(|P| + |Q|)|P - Q|^2, \\
|A(x,P) - A(x,Q)| &\leq c \varphi'(|P| + |Q|)|P - Q|,
\end{align*}$$

where $\varphi$ is a given Orlicz function. Moreover, we assume that the vector fields $A : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ and $b : \Omega \to \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ satisfy the following continuity and growth assumptions with respect to $x$:

$$\begin{align*}
|A(x,Q) - A(x_0,Q)| &\leq c|x-x_0|^\alpha_1 \varphi'(|Q|), \\
|b(x,Q)| &\leq c(\varphi'(|Q|) + g_1(x)), \\
|b(x,Q) - b(x_0,Q)| &\leq c|x-x_0|^\alpha_2 (\varphi'(|Q|) + g_2(x) + g_2(x_0)), \\
|b(x,P) - b(x,Q)| &\leq c \varphi'(|P| + |Q|) \left(\frac{|P-Q|}{|P| + |Q|}\right)^\alpha_3,
\end{align*}$$

with $\alpha_1, \alpha_2, \alpha_3 \in (0, 1]$ and suitable $g_1, g_2 : \Omega \to [0, \infty)$.

The standard examples for the Orlicz function $\varphi$ are

$$\varphi_1(t) = \int_0^t (\mu + s^2)^{\frac{p-2}{2}} s \, ds, \quad \varphi_2(t) = \int_0^t (\mu + s)^{p-2} s \, ds,$$

where $\mu \geq 0$. The $p$-Laplacian corresponds to the choice $\mu = 0$. Systems which such a type of growth conditions have been studied by many authors for special situations.

The first partial regularity results for non-linear elliptic systems were achieved by Morrey [21], followed by Giusti and Miranda [15] and Giusti [13]. The work has been continued for example by Evans [9], Giaquinta [10], Carozza, Fusco, Mingione [6], and by Duzaar and Grotowski [8].

Suppose that $u$ is a weak solution to (1.1) and let $\Sigma$ denote the set of singularities of $\nabla u$, see Section 5 for precise definition. In this situation we try to show that $\Sigma$ is reasonable small, i.e. that the Hausdorff dimension of $\Sigma$ is small. If $\alpha_1 = 1$ then we speak of a differentiable elliptic system. For differentiable systems and minimizers with $p = 2$ it is shown, e.g. in [10] and [14] that the Hausdorff dimension is strictly less than $n - 2$.

The nonlinear, differentiable case with $p$-growth (the case $\varphi_1$) has for example been considered by Acerbi and Fusco [1]. They show that the Hausdorff dimension of $\Sigma$ is strictly less or equal to $n - p$ for $1 < p < 2$. For $p \geq 2$ it can be seen by [9], [11], and [5] that the Hausdorff dimension is less or equal to $n - 2$. Let us point up here that the cases $1 < p \leq 2$ and $p \geq 2$ required different techniques in the mentioned papers. It is one of the main advantages of our approach that such distinction is not necessary anymore.

For non-differentiable systems for a long time it has only been know that $\Sigma$ has Lebesgue measure zero. So the question arose if it is possible gain more control of $\Sigma$ for non-differentiable systems. In particular, Giaquinta and Modica asked in their paper [12] and also in the book [10], pg. 191, wether the Hausdorff dimension of the singular set could be estimated. In his two articles [19] and [20]
Mingione gave the answer to this question in the case $p \geq 2$, i.e., that the dimension is always less than $n - 2\alpha$ if $u$ is Hölder continuous and $A$ is allowed to depend also on $u$. Further he could show that this result is still true in lower dimension ($n \leq 4$) if one drops the a priori Hölder continuity assumption. For higher dimensions he showed that the dimension is always less than $n$ which was not known even for the Lipschitz case $\alpha_1 = 1$. Our main motivation was to transfer these results to the case of arbitrary Orlicz function, thus including the full case $1 < p < \infty$. Hereby, it was of great importance to us that the used technique will not distinguish the cases $1 < p \leq 2$ and $p \geq 2$. By the difference quotient method Mingione shows that $V\nabla u$ is in the Sobolev-Slobodeckii $W^{2\beta}_p$ for any $\beta < \alpha$. In Mingione’s papers the estimates are actually carried out in Nikol’ski spaces and then at the end translated to fractional Sobolev spaces. Rather than estimating the $V\nabla u$ we prefer to estimate the natural quantity $V(\nabla u)$, which in the case of $\varphi_1$ is given by $V(\nabla u) = (\mu + |\nabla u|^2)^{\frac{\alpha}{\alpha - 1}} \nabla u$. Additionally, we choose $\Sigma$ to be the set of singularities of $V(\nabla u)$ instead of $\nabla u$. We will see that this is much more natural for the non-linear system (1.1). We show that $V(\nabla u)$ is locally in the Nikol’ski space $A^{\alpha,2}$. This will be proven by the difference quotient method. The estimate for the Hausdorff measure of $\Sigma(V(\nabla u))$ is then a consequence of this regularity information. We will show that the Hausdorff dimension of $\Sigma$ is less or equal to $n - 2\alpha$.

Since our approach works for arbitrary Orlicz functions, it especially works for the full range $1 < p < \infty$. Therefore, our technique is new even in the case of differentiable $A$ with no $x$-dependence of $A$.

Additionally, we derivate estimates of Cacciopoli and Gehring type for $V(\nabla u)$. The result is based on a new, generalized Poincaré inequality for arbitrary Orlicz functions. This inequality might be of independent interest.

Under similar assumption partial regularity can be proved, i.e. $V(\nabla u)$ is Hölder continuous on the complement of the singular set. This will be the content of a forthcoming paper.

2 Notation and Basic Properties

Let $\Omega \subset \mathbb{R}^n$ be an bounded, open domain. By $Q$ we will always denote a cube in $\mathbb{R}^n$ with sides parallel to the axis. We write $Q \Subset \Omega$ if the closure of $Q$ is contained in $\Omega$. Let $|Q|$ denote the volume and length$(Q)$ the side length of $Q$. For $f \in L^1(Q)$ we define

$$\int_Q f(x) \, dx := \frac{1}{|Q|} \int_Q f(x) \, dx.$$ 

By $kQ$, with $k > 0$, we denote the cube with the same center and $k$ times the side length. For functions $f, g$ on $\Omega$ we define $(f, g) := \int_Q f(x)g(x) \, dx$. For $a, b \in \mathbb{R}^n$ we denote by $[a, b]$ the straight line segment from $a$ to $b$. If $a \neq b$ we define $\int_a^b \cdots ds$ to be the mean average integral over the line $[a, b]$. For $U, W \subset \mathbb{R}^n$ we define $U + W := \{u + w : u \in U, w \in W\}$. We write $f \sim g$ iff there exist constants $c_0, c_1 > 0$, such that

$$c_0 f \leq g \leq c_1 f,$$
where we always indicate on what the constants may depend. Furthermore, we use $c$ (no index) as a generic constant, i.e. its value may change from line to line but does not depend on the important variables.

The following definitions and results are standard in the context of $N$–function (see e.g. [22]). A real function $\phi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is said to be an $N$–function if it satisfies the following conditions: There exists the derivative $\phi'$ of $\phi$. This derivative is right continuous, non-decreasing and satisfies $\phi'(0) = 0$ and $\phi'(t) > 0$ for $t > 0$. Especially, $\phi$ is convex.

We say that $\phi$ satisfies the $\Delta_2$–condition, if there exists $c_1 > 0$ such that for all $t \geq 0$ holds $\phi(2t) \leq c_1 \phi(t)$. By $\Delta_2(\phi)$ we denote the smallest constant $c_1$. Since $\phi(t) \leq \phi(2t)$ the $\Delta_2$ condition is equivalent to $\phi(2t) \sim \phi(t)$. For a family $\phi_n$ of $N$–functions we define $\Delta_2(\{\phi_n\}) := \sup_n \Delta_2(\phi_n)$.

By $L^\phi$ and $W^{1,\phi}$ we denote the classical Orlicz and Sobolev-Orlicz spaces, i.e. $f \in L^\phi$ iff $\int f(x) dx < \infty$ and $f \in W^{1,\phi}$ iff $\nabla f \in L^\phi$.

By $(\phi')^{-1} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ we denote the function

$$(\phi')^{-1}(t) := \sup \{u \in \mathbb{R}^{\geq 0} : \phi'(u) \leq t\}.$$ 

If $\phi'$ is strictly increasing then $(\phi')^{-1}$ is the inverse function of $\phi'$. Then $\phi^* : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with

$$\phi^*(t) := \int_0^t (\phi')^{-1}(s) ds$$

is again an $N$–function and $(\phi^*)'(t) = (\phi')^{-1}(t)$ for $t > 0$. It is the complementary function of $\phi$. Note that $(\phi^*)^* = \phi$. For all $\delta > 0$ there exists $c_3$ (only depending on $\Delta_2(\{\phi, \phi^*\})$) such that for all $t, u \geq 0$ holds

$$tu \leq \delta \phi(t) + c_3 \phi^*(u).$$  

This inequality is called Young’s inequality. For all $t \geq 0$

$$\frac{t}{2} \phi' \left( \frac{t}{2} \right) \leq \phi(t) \leq t \phi'(t),$$

$$\phi \left( \frac{\phi^*(t)}{t} \right) \leq \phi^*(t) \leq \phi \left( \frac{2 \phi^*(t)}{t} \right).$$

Therefore, uniformly in $t \geq 0$

$$\phi(t) \sim \phi'(t)t, \quad \phi^* \left( \phi'(t) \right) \sim \phi(t),$$

where the constants only depend on $\Delta_2(\{\phi, \phi^*\})$. If $\rho(t) = a \phi(bt)$ for some $a, b > 0$ and all $t \geq 0$, then

$$\rho^*(t) = a \phi^* \left( \frac{t}{ab} \right).$$

If $\phi$ and $\rho$ are $N$–functions with $\phi(t) \leq \rho(t)$ for all $t \geq 0$, then

$$\rho^*(t) \leq \phi^*(t)$$

for all $t \geq 0$.

In most parts of the paper we will assume that $\phi$ satisfies the following assumptions:
**Assumption 1** Let $\varphi$ be an $N$-function such that $\Delta_2(\{\varphi, \varphi^*\}) < \infty$. Further assume that $\varphi$ is $C^2$ on $(0, \infty)$ and uniformly in $t \geq 0$

$$\varphi'(t) \sim t \varphi''(t), \quad (2.6)$$

As already mentioned in the introduction we will assume that the vector field $A : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ satisfies the non-standard $\varphi$-growth condition, i.e.

$$\begin{align*}
(A(x, P) - A(x, Q)) : (P - Q) &\geq c \varphi''(|P| + |Q|) |P - Q|^2, \\
|A(x, P) - A(x, Q)| &\leq c \varphi''(|P| + |Q|) |P - Q|
\end{align*} \quad (2.7)$$

and the continuity and growth condition

$$|A(x, Q) - A(x_0, Q)| \leq c |x - x_0|^{\alpha_1} \varphi'(|Q|), \quad (2.8)$$

where $0 < \alpha_1 \leq 1$ and $\varphi$ satisfies Assumption 1.

It is interesting to know that for every $\varphi$ as in Assumption 1 there exists $A : \Omega \to \mathbb{R}^{N \times n}$ that satisfies (2.7). The construction of such $A$ can be found in Lemma 21 in the appendix. It is possible to multiply $A$ by some function $\mu : \Omega \to (0, \infty)$ which is uniformly $\alpha_1$-Hölder continuous and is bounded from above and below. Then (2.7) and (2.8) still hold.

**Remark 1** Our standard examples for $A$ and $\varphi$ are

$$A(x, Q) := \mu(x) |Q|^{p-2} Q, \quad \varphi'(t) := t^{\alpha - 1}$$

and

$$A(x, Q) := \mu(x) (1 + |Q|)^{p-2} Q, \quad \varphi'(t) := (1 + t)^{p-2} t,$$

where $1 < p < \infty$, $0 < \alpha_1 \leq 1$, and $\mu : \Omega \to (0, \infty)$ is $\alpha_1$-Hölder continuous and bounded from above and below.

For given $\varphi$ we define the $N$-function $\psi$ by

$$\psi'(t) := \left( \frac{\varphi'(t)}{t} \right)^{\frac{1}{\alpha}}. \quad (2.9)$$

It is shown in Lemma 25 that $\psi$ also satisfies Assumption 1 and uniformly in $t > 0$ holds $\psi''(t) \sim \sqrt[\alpha]{\varphi''(t)}$. As in Lemma 21 we define $\Psi : \mathbb{R}^{n \times n} \to \mathbb{R}^{\geq 0}$ by $\Psi(Q) := \psi(|Q|)$ and let $V(Q) := (\nabla_{N \times n} \Psi)(Q) = \psi'(|Q|) \frac{Q}{|Q|}$. From the same lemma it follows that (2.7) holds with $A$, $\varphi$ replaced by $V$, $\psi$.

**Remark 2** The examples given in Remark 1 correspond to

$$V(Q) := |Q|^\frac{p-2}{2} Q, \quad \psi'(t) := t^\frac{p}{2}$$

and

$$V(Q) := (1 + |Q|)^{\frac{p-2}{2}} Q, \quad \psi'(t) := (1 + t)^{\frac{p}{2}} t,$$

where $1 < p < \infty$. 
We introduce a family of $N$-function $\{\varphi_a\}_{a \geq 0}$ by $\varphi'(t)/t := \varphi'(a + t)/(a + t)$ which basically states $\varphi_a(t) \sim \varphi(t)$ uniformly in $a, t \geq 0$. The basic properties of $\varphi_a$ are given in the appendix, see Definition 22 and thereafter. The connection between $A$, $V$, and $\{\varphi_a\}_{a \geq 0}$ is best reflected in the following lemma.

**Lemma 3** Let $A, \varphi$ satisfy Assumption 1 and (2.7). Let $\psi, V$ be defined as in (2.9). Then

\[
(A(x, P) - A(x, Q)) \cdot (P - Q) \sim |V(P) - V(Q)|^2 \quad (2.10a)
\]

\[
\sim \varphi_P(|P - Q|), \quad (2.10b)
\]

\[
\sim |P - Q|^2 \varphi''(|P| + |Q|), \quad (2.10c)
\]

uniformly in $P, Q \in \mathbb{R}^{N \times n}$ and $x \in \Omega$. Moreover,

\[
A(x, Q) \cdot Q \sim |V(Q)|^2 \sim \varphi(|Q|) \quad (2.10d)
\]

uniformly in $Q \in \mathbb{R}^{N \times n}$ and $x \in \Omega$.

Note that if $\varphi''(0)$ does not exists, the expression in (2.10c) is continuously extended by zero for $|P| = |Q| = 0$.

The lemma will be proven in the appendix. The different representations of (2.10) will be useful at different stages of our proofs. The one with $A$ appears when we test the differential operator $-\text{div}(A(Vu))$ by a suitable test function. The one with $V$ is useful to write down information, since most of the information on $u$ will be expressed in information on $V(Vu)$. The representation with $\varphi_a$ simplifies the proofs. The function $V$ also appears in the study of minimizers of the form $\int \varphi(|Vu|) \, dx$.

For the right hand side of the system (1.1) we assume that the vector field $b : \Omega \to \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ satisfies the following continuity and growth assumptions with respect to $x$:

\[
|b(x, Q)| \leq c(\varphi'(|Q|) + g_1(x)), \quad (2.11a)
\]

\[
|b(x, Q) - b(x_0, Q)| \leq c|x - x_0|^{\alpha_2}(\varphi'(|Q|) + g_2(x) + g_2(x_0)), \quad (2.11b)
\]

\[
|b(x, P) - b(x, Q)| \leq c\varphi'(|P| + |Q|) \left(\frac{|P - Q|}{|P| + |Q|}\right)^{\alpha_3}, \quad (2.11c)
\]

with $\alpha_2, \alpha_3 \in (0, 1], g_1, g_2 : \Omega \to \mathbb{R}^{+}$, and $\varphi^*(|g_1|), \varphi^*(|g_2|) \in L^q$ for some $q > 1$. Again, it will be useful to introduce a suitable family of $N$–functions $\{\varphi_{a, \omega}\}_{a \geq 0}$ to clarify the natural choice of the growth condition (2.11c). Especially, let $\omega_3(t) := 1/(\alpha_3 + 1)t^{\alpha_3 + 1}$, i.e. $\omega_3'(t) = t^{\alpha_3}$ then (2.11c) can be rewritten as

\[
|b(x, P) - b(x, Q)| \leq c\varphi'_{|P|, \omega_3}(|P - Q|), \quad (2.12)
\]

where $\varphi_{a, \omega}$ is given in Definition 22. We will see later in the proof of Theorem 11 and Lemma 12 that this form of continuity condition is the natural one.
3 Caccioppoli Estimates and a Gehring Type Result

In the following assume that $u$ is a weak solution of system (1.1) in the sense that $u$ satisfies (1.1) in the distributional sense and that $Vu \in L^q(\Omega)$. In view of (2.10d) this is equivalent to $V(\nabla u) \in L^2(\Omega)$. We start with a lemma of Caccioppoli type. At this point we would like to mention that in order to keep to notations short we will often skip the explicit dependence on $x$. For example we will rather write $A(\nabla u)$ instead of $A(x, (\nabla u)(x))$. Nevertheless, $A$ will still depend on $x$.

**Theorem 4** Let $u$ be a weak solution of system (1.1). Then there exists $c > 1$ such that for all cubes $Q$ with $2Q \Subset \Omega$ holds

$$
\int_Q \varphi(\nabla u) dx \leq c \int_Q \varphi\left(\frac{|u - (u)_Q|}{R}\right) dx + c \int_Q \varphi^*(|g_1|) dx,
$$

(3.1)

where $R$ is the side length of the cube $Q$. The constant $c$ only depends on $\Delta_2[\{\Phi, \Phi^*\}]$ and the constants in (2.6), (2.7), (2.8), and (2.11).

**Proof** For fixed $Q$ and $R := \text{length}(Q)$ let $\eta \in C_0^\infty(2Q)$ be a cut-off function with $\chi_Q \leq \eta \leq \chi_{2Q}$ and $|\nabla \eta| \leq c/R$. We pick the test function $\xi := \eta^q(u - (u)_Q)$ and obtain

$$
\langle A(\nabla u), \eta^q \nabla u \rangle = -\langle A(\nabla u), q \eta^{q-1}(u - (u)_Q) \otimes (\nabla \eta) \rangle
+ \langle b(\nabla u), \eta^q(u - (u)_Q) \rangle.
$$

The exponent $q$ will be chosen as follows. By Lemma 31 there exist $\varepsilon, c_2 > 0$ with $\varphi(\lambda t) \leq c_2 \lambda^{1+\varepsilon} \varphi(t)$ uniformly in $t \geq 0$ and $\lambda \in [0, 1]$. We fix $q > 1$ large enough such that $(1 + \varepsilon)(q - 1) \geq q$. In particular, uniformly in $t \geq 0$

$$
\varphi(\eta^{q-1}t) \leq c_2 \eta^q \varphi(t).
$$

(3.2)

The monotonicity and growth conditions on $A$ and $b$ imply

$$
\int_{2Q} \eta^q \varphi(\nabla u) dx \leq c \int_{2Q} \eta^{q-1} \varphi'(\nabla u) \frac{|u - (u)_Q|}{R} dx
+ c \int_{2Q} \eta^q (\varphi'(\nabla u) + g_1)|u - (u)_Q| dx.
$$

According to Young’s inequality (6.26) we derive for $\varepsilon > 0$

$$
\int_{2Q} \eta^q \varphi(\nabla u) dx \leq c \int_{2Q} \varphi^*(\eta^{q-1} \varphi'(\nabla u)) dx + \varepsilon \int_{2Q} \eta^q \varphi(\nabla u) dx
+ c_\varepsilon \int_{2Q} \varphi\left(\frac{|u - (u)_Q|}{R}\right) dx + c_\varepsilon \int_{2Q} \varphi^*(|g_1|) dx,
$$

where we have used that $R \leq c(\Omega)$, since $\Omega$ is bounded. (This is the only place in the paper where we use the boundedness of $\Omega$.) Note that by (3.2) and (2.3)

$$
\varphi^*(\eta^{q-1} \varphi'(\nabla u)) \leq c \eta^q \varphi^*(\varphi'(\nabla u)) \sim \eta^q \varphi(\nabla u).
$$
Thus for small $\varepsilon > 0$ we deduce
\[
\int_Q \varphi(|\nabla u|) \, dx \leq c \int_{2Q} \varphi \left( \frac{|u - \langle u \rangle|_Q}{R} \right) \, dx + c \int_{2Q} \varphi^*(|g_1|) \, dx,
\]
where we have used $\eta^q \geq \chi_Q$. This the Theorem.

**Remark 5** It is easy to see that Theorem 4 and the results below remain valid if we use balls instead of cubes.

From Theorem 4 we want to derive an estimate of Gehring type, i.e., some reverse Hölder estimate. It is standard to use the ingenious lemma of Giaquinta and Modica:

**Proposition 6 (Giaquinta-Modica)** Let $Q_0 \subset \mathbb{R}^n$ be a cube, $G \in L^1(Q_0)$, and $H \in L^{n/(n-1)}(Q_0)$ for some $q_0 > 1$. Suppose that for some $\theta \in (0, 1)$, $c_1 > 0$, and all cubes $Q$ with $2Q \subset Q_0$
\[
\int_Q |G| \, dx \leq c_1 \left( \int_{2Q} |G^\theta \, dx \right)^{\frac{1}{\theta}} + \int_{2Q} |H| \, dx.
\]

Then there exist $q_1 > 1$ and $c_2 > 1$ such that $G \in L^{q_1}_{\text{loc}}(Q)$ and for all $q_2 \in [1, q_1]$
\[
\left( \int_Q |G|^{q_2} \, dx \right)^{\frac{1}{q_2}} \leq c_2 \int_{2Q} |G| \, dx + c_2 \left( \int_{2Q} |H|^{q_2} \, dx \right)^{\frac{1}{q_2}}.
\]

Another important tool in our proof will be the following generalization of the Poincaré’s inequality.

**Theorem 7 (Poincaré type)** Let $\varphi$ be an $N$--function with $\Delta_2(\{\varphi, \varphi^*\}) < \infty$. Further, let $Q \subset \mathbb{R}^n$ be some cube with side length $R$ and let $\omega \in L^\infty(Q)$ with $\omega > 0$ and $\int_Q \omega(x) \, dx = 1$. Then there exists $0 < \theta < 1$, which only depends on $\Delta_2(\{\varphi, \varphi^*\})$, and there exists $K > 0$, which only depends on $\Delta_2(\{\varphi, \varphi^*\})$ and $R^\theta \|\omega\|_{\infty}$, such that for all $v \in W^{1, \theta}(Q)$ holds
\[
\int_Q \varphi \left( \frac{|v - \langle v \rangle_\omega|}{R} \right) \, dx \leq K \left( \int_Q (\varphi(|\nabla v|))^\theta \, dx \right)^{\frac{1}{\theta}},
\]
where $\langle v \rangle_\omega := \int_Q v(x) \omega(x) \, dx$.

Note that for the special choice $\omega := |Q|^{-1} \chi_Q$ we have $\langle v \rangle_\omega = \langle v \rangle_Q$.

**Proof** Since $\Delta_2(\varphi^*) < \infty$ it follows from [17] (Lemma 1.2.2+1.2.3) that $\varphi^\theta$ is quasiconvex for some $1 - \frac{1}{\theta} < \theta < 1$, i.e. there exists an $N$–function $\rho$ with $\varphi^\theta \sim \rho$ and $\Delta_2(\{\rho, \rho^*\}) < \infty$. It is important to remark that $\theta$ and $\Delta_2(\{\rho, \rho^*\})$ only depend on $\Delta_2(\{\varphi, \varphi^*\})$. We deduce that $\varphi(\rho^{-1}(t)) \sim t^{\frac{1}{\theta}}$. Let $L := \int_Q \rho(\nabla v) \, dx$. 
If \( L = 0 \) then \( \mathbf{v} \) is constant on \( Q \) and there is nothing to show. So we assume that \( L > 0 \). From [18] (Lemma 1.50) we know that for almost all \( x \in Q \) holds
\[
|\mathbf{v}(x) - (\mathbf{v})_x| \leq c \int_Q \frac{|\nabla \mathbf{v}(y)|}{|x - y|^{n-1}} \, dy,
\]
where the constant only depends on \( R^n \|\omega\|_{\infty} \).

With (3.4) and \( \Lambda_2(\phi) < \infty \) we estimate
\[
(I) := \int_Q \phi \left( \frac{|\mathbf{v} - (\mathbf{v})_x|}{R} \right) \, dx \leq c \int_Q \phi \left( \int_Q |\nabla \mathbf{v}(\xi)| \, |x - \xi|^{n-1} \, d\xi \right) \, dx.
\]

Since \( \int_Q R^{-1} |x - \xi|^{-n} \, dx \leq c \) independent of \( Q \) and \( x \in Q \), we can apply Jensen’s inequality to the convex function \( \rho \) and the measure \( R^{-1} |x - \xi|^{-1} \, d\xi \). This implies
\[
(I) \leq c \int_Q \phi \circ R^{-1} \circ \left( \int_Q \rho \left( |\nabla \mathbf{v}(\xi)| \right) \, R^{-1} |x - \xi|^{-n} \, d\xi \right) \, dx
\leq c \int_Q \left( \int_Q \rho \left( |\nabla \mathbf{v}(\xi)| \right) \, R^{-1} |x - \xi|^{-n} \, d\xi \right)^{\frac{1}{\theta}} \, dx
\leq c R^{-1/\theta} \int_Q L^{1/\theta} R^{n/\theta} \left( \int_Q L^{-1} \rho \left( |\nabla \mathbf{v}(\xi)| \right) \, |x - \xi|^{-n} \, d\xi \right)^{\frac{1}{\theta}} \, dx,
\]
where we have used \( \Lambda_2(\{\rho, \rho^*\}) < \infty \). Now Jensen’s inequality applied to the convex function \( t \mapsto t^{1/\theta} \) and the measure \( L^{-1} \rho (|\nabla \mathbf{v}(\xi)|) \, d\xi \) gives
\[
(I) \leq c R^{(n-1)/\theta} \int_Q L^{1/\theta} \int_Q L^{-1} \rho \left( |\nabla \mathbf{v}(\xi)| \right) \, \left( |x - \xi|^{-n} \right)^{\frac{1}{\theta}} \, d\xi \, dx
\leq c R^{(n-1)/\theta} L^{1/\theta - 1} \int_Q \rho \left( |\nabla \mathbf{v}(\xi)| \right) \, d\xi \, R^{(n-1)/\theta},
\]
which is possible since \( \frac{1-n}{\theta} > -n \). By definition of \( L \)
\[
(I) \leq c \left( \int_Q \rho \left( |\nabla \mathbf{v}(\xi)| \right) \, d\xi \right)^{\frac{1}{\theta}} \leq c \left( \int_Q \rho \left( |\nabla \mathbf{v}(\xi)| \right) \, d\xi \right)^{\frac{1}{\theta}}.
\]

This proves the theorem.

Remark 8 Theorem 7 is probably well-known among experts, but we could not find a reference. A proof of the simplified case \( \theta = 1 \) and \( \omega = |Q|^{-1} \chi_Q \) can be found in [4]. Nevertheless, we need the sharper version with \( \theta < 1 \) in Theorem 9 in order to apply Proposition 6. Moreover, we will need the version with general \( \omega \) in a forthcoming article.
We are now able to prove the reverse Hölder estimate.

**Theorem 9** Let $u$ be a weak solution of system (1.1). Then there exists $q_2 > 1$ and $c > 1$ such that for all cubes $Q$ with $2Q \subset \Omega$ and all $q \in [1,q_1]$ holds

$$
\left( \int_Q |V(\nabla u)|^{2q} \, dx \right)^{\frac{1}{q}} \leq c \int_{2Q} |V(\nabla u)|^2 \, dx + c \left( \int_{2Q} (\varphi^+(g_1))^q \, dx \right)^{\frac{1}{q}},
$$

(3.5a)

Especially, by (2.10d) we have $\varphi(|\nabla u|) \in L_{loc}^{q_1}(\Omega)$. The constants $c$ and $q_1$ only depend on $\Delta_2(\{\varphi, \varphi^+\})$ and the constants in (2.6), (2.7), (2.8), and (2.11).

**Proof** Due to Theorem 4 we have

$$
\int_Q \varphi(|\nabla u|) \, dx \leq c \int_{2Q} \varphi\left( \frac{|u - \langle u \rangle_Q|}{R} \right) \, dx + c \int_{2Q} \varphi^+(g_1) \, dx.
$$

By Theorem 7 there exists $\theta \in (0,1)$ only depending on $\Delta_2(\{\varphi, \varphi^+\})$ such that

$$
\int_Q \varphi(|\nabla u|) \, dx \leq c \left( \int_{2Q} (\varphi(|\nabla u|))^{\theta} \, dx \right)^{\frac{1}{\theta}} + c \int_{2Q} \varphi^+(g_1) \, dx.
$$

From Proposition 6 we deduce that there exists $q_1 > 1$ such that for all $q \in [1,q_1]$

$$
\left( \int_Q (\varphi(|\nabla u|))^q \, dx \right)^{\frac{1}{q}} \leq c \int_{2Q} \varphi(|\nabla u|) \, dx + c \left( \int_{2Q} (\varphi^+(g_1))^q \, dx \right)^{\frac{1}{q}}.
$$

This and (2.10d) proves the theorem.

**Remark 10** Note that similar results regarding higher integrability have been proved in [7] by A. Chianchi and N. Fusco.

## 4 Modified Difference Quotient Method

In this section we derive higher regularity of the solution $u$, especially we show that $V(\nabla u)$ is locally in the Nikol’skii space $N^{\alpha,2}$. To prove this we use a modified version of the difference quotient method. Instead of plain differences we will at a certain stage consider averages of differences. Let us introduce the notations: For $x,s \in \mathbb{R}^n$ we define

$$
T_s(x) := x + s, \quad (\tau_s f)(x) := f(x + s) - f(x).
$$

The main theorem is the following.
Theorem 11 Let $u$ be a weak solution of system (1.1). Then there exists $c_3 > 0$ such that the following holds: If $Q \subset \Omega$ is a cube with $2Q \Subset \Omega$ and if $h \in \mathbb{R}^n \setminus \{0\}$ with $|h| \leq R$ then
\[
\int_Q |\tau_h V(\nabla u)|^2 \, dx \leq c_3 \beta(R, |h|) \left( \int_{2Q} |V(\nabla u)|^2 \, dx + \int_{2Q} \varphi^*(g_2) \, dx \right),
\]
where $\beta(R, |h|) := \frac{|h|^2}{R^2} + \frac{|h|^{1+a_1}}{R} + |h|^{2a_2} + |h|^{1+a_2}$. The constant $c_3$ only depend on $\Delta_2(\{\phi, \phi^*\})$ and the constants in (2.6), (2.7), (2.8), and (2.11).

We split the proof in two parts and begin with the following lemma.

Lemma 12 Let $u$ be a weak solution of system (1.1). Then for every $\delta > 0$ there exists $c_\delta > 0$ such that the following holds: If $Q \subset \Omega$ is a cube with $4Q \Subset \Omega$ and if $h, s \in \mathbb{R}^n \setminus \{0\}$ with $|s| \leq |h| \leq R$ then
\[
\int_Q |\tau_s V(\nabla u)|^2 \, dx \leq \delta \frac{|s|}{|h|} \int_0^s \int_{2Q} |\tau_h V(\nabla u)|^2 \, dx \, d\lambda \nabla\beta(R, |s|) \int_{2Q} |V(\nabla u)|^2 \, dx + c |s|^{1+a_2} \int_{4Q} \varphi^*(g_2) \, dx
\]
and
\[
\int_0^h \int_Q |\tau_s V(\nabla u)|^2 \, dx \, d\lambda \leq \delta \int_0^h \int_{2Q} |\tau_h V(\nabla u)|^2 \, dx \, d\lambda + c_\delta \beta(R, |h|) \int_{4Q} |V(\nabla u)|^2 \, dx + c |h|^{1+a_2} \int_{4Q} \varphi^*(g_2) \, dx,
\]
where $\beta(R, |h|)$ is defined as in Theorem 11.

Proof As in the classical approach we first apply $\tau_s$ to our system (1.1). Let $Q, R,$ and $s$ be as required and $\xi \in C^0(\Omega)$. Then
\[
\langle \tau_s (A(\nabla u)), \nabla \xi \rangle = \langle \tau_s (b(\nabla u)), \xi \rangle.
\]
(Recall that $\langle f, g \rangle := \int_{\Omega} f(x)g(x) \, dx$.) As before we skipped the explicit dependence on $x$ in order to keep the notations short, but $A$ and $b$ nevertheless depend on $x$. As in the proof of Theorem 9 we can choose with the help of (6.25) some $q > 1$ and $c_2 > 0$ such that
\[
\varphi_q(\eta^{q-1} \xi) \leq c_2 \eta^q \varphi_q(\xi) \tag{4.4}
\]
uniformly in $t, a \geq 0$. Let $\eta \in C^\infty_c$ be a cut off function with $\chi_Q \leq \eta \leq \chi_{2Q}$ and $|\nabla \eta| \leq c/R$. Then we use the test function $\xi := \eta^q \tau_s u$. We get
\[
\langle \tau_s (A(\nabla u)), \nabla (\eta^q \tau_s u) \rangle = \langle \tau_s (b(\nabla u)), \eta^q \tau_s u \rangle \tag{4.5}
\]
Everything will be derived from this equation. We define
\[ \mathcal{A}_s(x) := A(x, \nabla u(x + s)) - A(x, \nabla u(x)), \]
\[ \mathcal{B}_s(x) := A(x + s, \nabla u(x + s)) - A(x, \nabla u(x) + s), \]
\[ \mathcal{C}_s(x) := b(x, \nabla u(x + s)) - b(x, \nabla u(x)), \]
\[ \mathcal{D}_s(x) := b(x + s, \nabla u(x + s)) - b(x, \nabla u(x) + s), \]
then
\[ \left( \tau_s(A(\nabla u)) \right)(x) = \mathcal{A}_s(x) + \mathcal{B}_s(x), \]
\[ \left( \tau_s(b(\nabla u)) \right)(x) = \mathcal{C}_s(x) + \mathcal{D}_s(x). \]

Now (4.5) reads
\[ \langle \mathcal{A}_s + \mathcal{B}_s, \nabla (\eta \tau_s u) \rangle = \langle \mathcal{C}_s + \mathcal{D}_s, \eta \tau_s u \rangle. \] (4.6)

Let (I), (II), (III), and (IV) be the four summands in (4.6). Let us collect the fundamental estimates for \( \mathcal{A}_s, \mathcal{B}_s, \mathcal{C}_s, \) and \( \mathcal{D}_s \):
\[ \mathcal{A}_s \cdot \tau_s \nabla u \approx \varphi_{\nabla u}(|\tau_s \nabla u|) \sim |\tau_s \nabla u|^2, \] (4.7a)
\[ |\mathcal{A}_s| \leq c \varphi_{\nabla u}(|\tau_s \nabla u|), \] (4.7b)
\[ |\mathcal{B}_s| \leq c |s|^{\alpha_0} \varphi'(|\nabla u \circ \mathcal{T}_s|), \] (4.7c)
\[ |\mathcal{C}_s| \leq c \varphi_{\nabla u, \alpha_0}(|\tau_s \nabla u|), \] (4.7d)
\[ |\mathcal{D}_s| \leq c |s|^{\alpha_0} \varphi'(\varphi'(|\nabla u \circ \mathcal{T}_s|) + g_2 \circ \mathcal{T}_s), \] (4.7e)

where \( \alpha_0 \) is defined by \( \alpha'_0(t) := t^{-1}. \) These inequalities follow directly from the assumptions (2.7) and (2.8) on \( A, \) the assumptions (2.11) and (2.12) on \( b, \) and the fundamental lemma 3 which provides the connection of \( A, V, \varphi, \) and \( \{ \varphi_u \}_{u \geq 0}. \) As an example we will derive (4.7a) and (4.7b) in detail:
\[ \mathcal{A}_s(x) \cdot (\tau_s \nabla u)(x) \]
\[ = \left( A(x, \nabla u(x + s)) - A(x, \nabla u(x)) \right) \cdot (\tau_s \nabla u)(x) \]
\[ \approx \varphi''(|\nabla u(x + s)| + |\nabla u(x)|) \cdot |\tau_s \nabla u(x)|^2 \quad \text{by (2.7)} \]
\[ \approx \varphi_{\nabla u}(x) \cdot |(\tau_s \nabla u)(x)| \quad \text{by Lemma 24} \]
\[ \approx |(\tau_s \nabla u)(x)|^2 \quad \text{by (2.10)}, \]
\[ |\mathcal{A}_s(x)| = |A(x, \nabla u(x + s)) - A(x, \nabla u(x))| \]
\[ \leq c \varphi''(|\nabla u(x + s)| + |\nabla u(x)|) \cdot |(\tau_s \nabla u)(x)| \quad \text{by (2.7)} \]
\[ \sim c \varphi_{\nabla u}(x) \cdot |(\tau_s \nabla u)(x)| \quad \text{by Lemma 24} \]

We split \( \nabla \xi \) into \( \nabla \xi = \eta \tau_s \nabla u + q \eta^{-1} (\tau_s u) \circ \nabla \eta. \) Then
\[ (I) = \langle \mathcal{A}_s, \eta \tau_s \nabla u \rangle + \langle \mathcal{A}_s, q \eta^{-1} (\tau_s u) \circ \nabla \eta \rangle =: (I_1) + (I_2). \]
Analogously, we split \((II)\) into \((II_1) + (II_2)\). By (4.7a)

\[
(II_1) = \langle \omega, \eta^q \tau_s \nabla u \rangle \sim \int_{2Q} \eta^q |\tau_s \nabla (\nabla u)|^2 \, dx.
\] (4.8)

This term is the good term while all other terms have to be controlled. Using (4.7b) and the estimates on \(\nabla \tau\) we estimate (II) .

\[
(II_2) \leq c \int_{2Q} \eta^q \varphi'_{\nabla u} (|\tau_s \nabla u|) \frac{|\tau_s u|}{R} \, dx.
\] (4.9)

Since our good term \((II_1)\) only carries information on \(\nabla u\) we have to find a way to estimate \(\tau_s u\) in terms of \(\nabla u\). The representation

\[
(\tau_s u)(x) = \int_0^x \sum_i (\partial_i u)(x + \lambda) \frac{s_i}{|s|} \, d\lambda
\]

provides the estimates

\[
|((\tau_s u)(x)| \leq |s| \int_0^x |(\nabla u \circ T_\lambda)(x)| \, d\lambda.
\] (4.10)

From (4.9) and (4.10) we get

\[
(II_2) \leq c \int_{2Q} \eta^q \varphi'_{\nabla u} (|\tau_s \nabla u|) \int_0^x \frac{|s|}{R} |\nabla u \circ T_\lambda| \, d\lambda \, dx.
\] (4.11)

Let us define

\[
(J) := \eta^q \varphi'_{\nabla u} (|\tau_s \nabla u|) \frac{|h|}{R} |\nabla u \circ T_\lambda|
\]

Please notice the \(|h|/R\) instead of \(|s|/R\). We will need the remaining factor \(|s|/|h|\) later. From Lemma 29 (with \(a = \nabla u, b = \nabla u \circ T_\lambda, \omega(t) = \frac{1}{2} t^2,\) and \(e = \nabla u \circ T_\lambda\)) we deduce

\[
(J) \leq c \eta^q \left( \varphi'_{\nabla u \circ T_\lambda} (|\tau_s - \lambda \nabla u \circ T_\lambda|) + \varphi'_{\nabla u \circ T_\lambda} (|\tau_s \nabla u|) \right) \frac{|h|}{R} |\nabla u \circ T_\lambda|.
\]

Now, Young’s inequality (6.27), (4.4), (6.22), and (2.10d) imply

\[
(J) \leq \delta \varphi'_{\nabla u \circ T_\lambda} (|\tau_s - \lambda \nabla u \circ T_\lambda|) + c_2 \delta |\nabla u \circ T_\lambda| + c_2 \delta \eta^q |\nabla u \circ T_\lambda| + c_2 \delta \eta^q |\nabla u \circ T_\lambda| + c_2 \delta |\nabla u \circ T_\lambda| + c_2 \delta |\nabla u \circ T_\lambda| + c_2 \delta |\nabla u \circ T_\lambda|
\]

\[
\sim \delta \eta^q |\tau_s - \lambda \nabla (\nabla u) \circ T_\lambda|^2 + \delta \eta^q |\tau_s \nabla (\nabla u)|^2 + c_2 \delta \frac{|h|^2}{R^2} |\nabla u \circ T_\lambda|.
\]
Let us combine this with (4.11) then
\[
(I_2) \leq \delta \int_{2Q} \eta^q \frac{|s|}{|h|} \int_0^s |\tau_{s-l} \nabla (\nabla u) \circ T_\lambda|^2 + |\tau_\lambda \nabla (\nabla u)|^2 \, d\lambda \, dx
+ c_\delta \frac{|h|^2}{R^2} \int_{2Q} |\nabla (\nabla u \circ T_\lambda)|^2 \, dx,
\]
(4.12)

Note that in general for $|s| \leq |h| \leq R$
\[
\int_{2Q} \int_0^s |(f \circ T_\lambda)(x)| \, d\lambda \, dx \leq c \int_{4Q} |f(x)| \, dx,
\]
(4.13)
\[
\int_{2Q} \int_0^s |(\tau_{s-l} f \circ T_\lambda)(x)| \, d\lambda \, dx \leq c \int_{4Q} \int_0^s |\tau_\lambda f(x)| \, d\lambda \, dx.
\]
(4.14)

This, (4.12), and $|s| \leq |h|$ imply
\[
(I_2) \leq \delta \int_{2Q} \eta^q \frac{|s|}{|h|} \int_0^s |\tau_\lambda \nabla (\nabla u)|^2 \, d\lambda \, dx + c_\delta \frac{|h|^2}{R^2} \int_{2Q} |\nabla (\nabla u)|^2 \, dx.
\]
(4.15)

We estimate $(II_1)$ with (4.7c) and $\chi_Q \leq \eta \leq \chi_{2Q}$
\[
(II_1) = \langle \mathcal{B}_s, \eta^q \tau_\lambda \nabla u \rangle \leq c \int_{2Q} \eta^q |s|^a \phi'(|\nabla u \circ T_\lambda|) \tau_\lambda \nabla u \, dx.
\]

Note that by Young’s inequality (6.27)
\[
|s|^a \phi'(|\nabla u \circ T_\lambda|) \tau_\lambda \nabla u
\leq \delta \phi_{|\nabla u \circ T_\lambda|} + c_\delta \phi_{|\nabla u \circ T_\lambda|} \left(|s|^a \phi'(|\nabla u \circ T_\lambda|)\right)
\sim \delta |\tau_\lambda \nabla (\nabla u)|^2 + c_\delta |s|^{2a} \phi(|\nabla u \circ T_\lambda|)
\]
by (2.10), (6.23).

In particular, with (2.10d)
\[
(II_1) \leq \delta \int_{2Q} \eta^q |\tau_\lambda \nabla (\nabla u)|^2 \, dx + c_\delta |s|^{2a} \int_{3Q} |\nabla (\nabla u)|^2 \, dx.
\]
(4.16)

We estimate $(II_2)$ with (4.7c) and (4.10)
\[
(II_2) = \langle \mathcal{B}_s, q \eta^{q-1} (\tau_\lambda u) \otimes \nabla \eta \rangle
\leq c \int_{2Q} |s|^a \phi'(|\nabla u \circ T_\lambda|) \frac{|s|}{R} \int_0^s |\nabla u \circ T_\lambda| \, d\lambda \, dx.
\]
By Young’s inequality (6.26b), (4.13), and (2.10d)

$$ (I_2) \leq c \int \frac{|s|^{1+\alpha_1}}{R} \left( \varphi(|\nabla u \circ T_\lambda|) + \int_0^s \varphi(|\nabla u \circ T_\lambda|) \, d\lambda \right) \, dx $$

$$ \leq c \frac{|s|^{1+\alpha_1}}{R} \int_{4Q} |V(\nabla u)|^2 \, dx. \tag{4.17} $$

We now come to (III). By (4.7d) and (4.10)

$$ (III) = \langle \varepsilon, \eta^q \tau u \rangle $$

$$ \leq c \int_{2Q} \eta^q \phi_{[\bar{\nu}, \bar{\omega}]_0}(|\tau u \nabla u|) |s| \int_0^s |\nabla u \circ T_\lambda| \, d\lambda \, dx. $$

Analogously to the term (J) above we estimate with (6.19)

$$ (J_2) := \phi_{[\nu, \omega]^0}(|\tau u \nabla u|) |h| |\nabla u \circ T_\lambda| $$

$$ \leq \left( \phi_{[\nu, \omega]^0}(|\tau_{\alpha} - \lambda u \nabla u \circ T_\lambda|) + \phi_{[\nu, \omega]^0}(|\tau_{\alpha} \nabla u|) \right) |h| |\nabla u \circ T_\lambda|. $$

Define the N-functions $\sigma$ and $\kappa$ by $\sigma'(t) := t^{\frac{\omega_2}{\omega_3}}$, $\kappa'(t) := t$. Then $\kappa'(1) = \sigma'(1) = \omega_2(1) = 1$, $\sigma(t) \sim t^{\frac{\omega_2}{\omega_3}}$, $\kappa(t) \sim t^{\frac{\omega_3}{\omega_2}}$, and $\sigma'(\omega_2(t)) \sim t^2 \sim \kappa(t)$. Particularly, $\sigma, \kappa, \omega_3$ satisfy the assumptions of Lemma 34. By Young’s inequality (6.31), $\eta_{n, \kappa} = \eta_\kappa$ and (2.10)

$$ (J_2) \leq \delta \phi_{[\nu, \omega]^0}(|\tau_{\alpha} - \lambda u \nabla u \circ T_\lambda|) + \delta \phi_{[\nu, \omega]^0}(|\tau_{\alpha} \nabla u|) $$

$$ + c_\delta \phi_{[\nu, \omega]^0} |h| |\nabla u \circ T_\lambda| $$

$$ \sim \delta |\tau_{\alpha} - \lambda u \nabla u \circ T_\lambda|^2 + \delta |\tau_{\alpha} \nabla u|^2 + c_\delta |h| \sigma(|h|) \varphi(|\nabla u \circ T_\lambda|). $$

$$ \leq c \delta |\tau_{\alpha} - \lambda u \nabla u \circ T_\lambda|^2 + \delta |\tau_{\alpha} \nabla u|^2 + c_\delta |h|^{\frac{2}{\omega_3}} |\nabla u \circ T_\lambda|^2. $$

Therefore

$$ (III) \leq \delta \int_{2Q} \eta^q \left\{ \int_0^s \frac{|s|^{1+\alpha_2}}{|h|} |\tau_{\alpha} \nabla u|^2 \, d\lambda \, dx \right\} + c_\delta c^{\frac{2}{\omega_3}} \int_{4Q} |V(\nabla u)|^2 \, dx, \tag{4.18} $$

where we have used (4.13) and (4.14) once more. We finally get to last term (IV).

With (4.7e), (4.10), (4.13), and (2.10d)

$$ (IV) = \langle \eta_{\tau}, \eta^q \tau u \rangle $$

$$ \leq c \int_{2Q} |s|^{1+\alpha_2} \left( \varphi(|\nabla u \circ T_\lambda|) + g_2 \circ T_\lambda \right) \int_0^s |\nabla u \circ T_\lambda| \, d\lambda \, dx $$

$$ \leq c |s|^{1+\alpha_2} \int_{2Q} \phi(|\nabla u \circ T_\lambda| + \varphi^{(g_2)} + \varphi^{(g_2 \circ T_\lambda)} \, d\lambda \, dx \tag{4.19} $$

$$ \leq c |s|^{1+\alpha_2} \left( \int_{4Q} |V(\nabla u)|^2 \, dx + \int_{4Q} \varphi^{(g_2)} \varphi^{(g_2 \circ T_\lambda)} \, dx \right). $$
If we combine all estimates (4.8), (4.15), (4.16), (4.17), (4.18), and (4.19), apply (4.13) to all terms involving $T_i$, and divide by $|Q|$ we get (4.2). Note that for any integrable function $k : \mathbb{R}^n \to \mathbb{R}$ by Fubini holds
\[
\int_0^h \frac{|s|}{|h|} \int_0^x |k(\lambda)| \, d\lambda \, ds = \int_0^h \frac{1}{|h|} \int_0^h |k(\lambda)| \, d\lambda \leq \int_0^h |k(\lambda)| \, d\lambda.
\] (4.20)

Thus (4.3) follow from (4.2) by application of $\int_0^h ds$. This proves the Lemma.

We are able to get rid of the first term on the right hand side in (4.3) with a Giaquinta-Modica type lemma.

**Lemma 13** Let $\gamma_1, \ldots, \gamma_M : (0, \infty) \times (0, \infty) \to [0, \infty)$ be such that $\gamma_m(R, |h|)$, $m = 1, \ldots, M$, is non-decreasing in $R$ and $|h|$. Let $v \in L^1_{\text{loc}}(\Omega)$, $w_1, \ldots, w_M \in L^1_{\text{loc}}(\Omega)$ be such that the following holds: For every $\delta > 0$ there exists $c_\delta > 0$ such that for every cube $Q \subset \Omega$ with side length $R$ and $4Q \Subset \Omega$ and every $h \in \mathbb{R}^n \setminus \{0\}$ with $|h| \leq R$ holds
\[
\int_0^h \int_Q |\tau_{\lambda} v|^2 \, dx \, ds \leq \delta \int_0^h \int_{2Q} |\tau_{\lambda} v|^2 \, dx \, ds + c_\delta \sum_{m=1}^M \gamma_m(R, h) \int_{4Q} |w_m| \, dx.
\] (4.21)

Then there exists $N_2 = N_2(n)$ and $\bar{c} > 0$ such that for every cube $Q_0 \subset \Omega$ with side length $R_0$ and $5Q_0 \Subset \Omega$ and for every $h_0 \in \mathbb{R}^n \setminus \{0\}$ with $|h_0| \leq \frac{R}{10}$ holds
\[
\int_{Q_0} |\tau_{h_0} v|^2 \, dx \leq \bar{c} \sum_{m=1}^M \gamma_m(N_2 R_0, h_0) \int_{5Q_0} |w_m| \, dx.
\]

**Proof** Let $Q_0$, $R_0$, and $h_0$ be as specified and let $Q_0 := 5Q_0$. We construct a family $\{W_j\}_{j \geq 1}$ of cubes in the following way:

(a) Split the set $5Q_0$ into $2^n$ equivalent cubes. Take these $2^n$ cubes as our initial family of cubes. In particular, $Q_0$ is contained in this family.

(b) Replace any cube $Q$ of the family which does not satisfy $4Q \subset \Omega_0$ into $2^n$ equivalent cubes. Repeat this step recursively.

Then we obtain a family of cubes which we denote by $\{W_j\}_{j \geq 1}$ with the following properties:

(i) $Q_0 = \bigcup W_j$ up to a set of measure zero.

(ii) $Q_0 = \bigcup 4W_j$.

(iii) The $W_j$, $j \geq 1$, are pairwise disjoint.

(iv) $Q_0 \in \{W_j\}$.

(v) There exists $N_1 = N_1(n) \in \mathbb{N}$ such that $\mathcal{N}(j) \leq N_1$ for all $j \in \mathbb{N}$, where $\mathcal{N}(j) := \# \{k : 4W_k \cap W_j \neq \emptyset\}$.
(vi) There exists $N_2 = N_2(n) \in \mathbb{N}$ such that $\frac{1}{N_2} R_k \leq R_j \leq N_2 R_k$ for every $k \in N(j)$, where $R_j$ is the side length of $W_j$.

Set $h_j := \frac{R_j}{h_0}$ and $\omega_j := \left(\frac{R_j}{h_0}\right)^2$. Especially, $\frac{h_j}{h_0} = \frac{h_0}{h_0}$. We apply (4.21) for every $W_j$ and $h_j$, multiply the result by $|W_j| \omega_j$ and sum up. We obtain

$$
\sum_j \omega_j \int_0^{h_j} \tau_j v_j^2 dx ds \leq \delta \sum_j \omega_j \int_0^{h_j} \tau_j v_j^2 dx ds + c_\delta \sum_{m=1}^M \gamma_m(R_j, |h_j|) \sum_j \omega_j \int_{4W_j} |w_m| dx
$$

(4.22)

Note that by triangle inequality

$$
\int_0^{h_j} \int_{2W_j} |\tau_j v_j|^2 dx ds \leq \int_0^{h_j} \int_{2W_j} N_2 \sum_{j=1}^{N_2} |\tau_j v_j \circ T_{N_2}^j|^2 dx ds
$$

$$
\leq N_2^2 \int_0^{h_j} \int_{4W_j} |\tau_j v_j|^2 dx ds
$$

$$
= N_2^2 \int_0^{h_j} \int_{4W_j} |\tau_j v_j|^2 dx ds.
$$

This and $4W_j \subset \bigcup_{k \in N(j)} W_k$ implies

$$
(I) \leq \delta N_2^2 \sum_k \sum_{j \in N(k)} \omega_j \int_0^{h_j/N_2} |\tau_j v_j|^2 dx ds.
$$

From (vi) we deduce $h_j \leq N_2 h_k$ and $\omega_j \leq N_2^2 \omega_k$, so

$$
(I) \leq \delta N_2^5 \sum_k \sum_{j \in N(k)} \omega_k \int_0^{h_j} |\tau_j v_j|^2 dx ds
$$

(4.23)

$$
\leq \delta c N_1 N_2^5 \sum_k \omega_k \int_0^{h_k} |\tau_j v_j|^2 dx ds.
$$

Analogously, we have

$$
(II) \leq c_\delta c N_1 N_2^5 \sum_{m=1}^M \gamma_m(N_2 R_k, N_2 h_k) \sum_k \omega_k \int_{W_k} |w_m| dx.
$$

(4.24)
If we combine (4.22), (4.23), and (4.24) and absorb (I) for small $\delta > 0$ on the left-hand side then with $\tilde{c} = \tilde{c}(N_1N_2)$

$$
\sum_j \omega_j \int_0^{h_j} |\nabla u|^2 dx ds \leq \tilde{c} \sum_{m=1}^M \gamma_m(N_2 R_k, N_2 h_k) \int_{W_k} |w_m| dx \\
\leq \tilde{c} \sum_{m=1}^M \gamma_m(N_2 R_k, N_2 h_k) \int_{6Q_0} |w_m| dx,
$$

where we have used $\omega_k \leq 1$ and (ii). Since $Q_0 \in \{W_j\}$ by (iv) and $\omega_0 = 1$ we get

$$
\int_0^{h_0} \int_{Q_0} |\nabla u|^2 dx ds \leq \tilde{c} \sum_{m=1}^M \gamma_m(N_2 R_k, N_2 h_k) \int_{6Q_0} |w_m| dx.
$$

This proves the lemma.

We are now prepared to prove Theorem 11.

**Proof (Proof of Theorem 11)** Let $Q, R, h$ be as specified. From (4.3) we know that the requirements of Lemma 13 are satisfied with

$$
\begin{align*}
\gamma_1(R, |h|) &:= \beta(R, |h|), \quad w_1 := \varphi(|\nabla u|), \\
\gamma_2(R, |h|) &:= \beta(R, |h|), \quad w_2 := \varphi^*({g_2}).
\end{align*}
$$

Thus Lemma 13 and $\gamma_2(N_2 R, N_2 |h|) \leq c \gamma_2(R, |h|)$ implies

$$
\int_0^h \int_Q |\nabla u|^2 dx ds \leq c \beta(R, |h|) \int_{6Q} |\nabla u|^2 dx + c |h|^{1+\alpha_2} \int_{6Q} \varphi^*({g_2}) dx, \tag{4.25}
$$

We use (4.25) to estimate the first term on the right-hand side of (4.2). We get

$$
\int_Q |\nabla u|^2 dx \leq c \beta(R, |h|) \int_{20Q} |\nabla u|^2 dx + c |h|^{1+\alpha_2} \int_{4Q} \varphi^*({g_2}) dx.
$$

This proves Theorem 11.

### 5 Dimension of the Singular Set

For a function $f \in L^{1,\infty}_{\text{loc}}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ open we define the singular sets

$$
\begin{align*}
\Sigma_1(f) := \left\{ x \in \Omega : \liminf_{\rho \searrow 0} \int_{B_\rho(x)} |f - \{f\}_{B_\rho(x)}| dy > 0 \right\}, \\
\Sigma_2(f) := \left\{ x \in \Omega : \# \lim_{\rho \searrow 0} \{f\}_{B_\rho(x)} \right\} \cup \left\{ x \in \Omega : \limsup_{\rho \searrow 0} |\{f\}_{B_\rho(x)}| = \infty \right\},
\end{align*}
$$
where \( B_p(x) \) is a ball centered at \( x \) with radius \( p \). Further, define \( \Sigma(f) := \Sigma_1(f) \cup \Sigma_2(f) \). By \( \mathcal{H}^{\beta} \) we denote the \( \beta \)-dimensional Hausdorff measure. To estimate the Hausdorff dimension of \( \Sigma(f) \) we will need the following theorem.

**Theorem 14** Let \( \Omega \subset \mathbb{R}^n \) be open and let \( 0 < \alpha \). Assume that \( f \in \mathcal{N}^{p, \alpha}(\Omega) \). Especially, \( f \in L^p(\Omega) \) and there exists \( c > 0 \) such that for any \( \tilde{\Omega} \subset \Omega \) and all \( 0 < h < \text{dist}(\tilde{\Omega}, \partial \Omega) \) holds

\[
\| \tau_h f \|_{L^p(\tilde{\Omega})} \leq c |h|^\alpha.
\]

Then for any \( \beta > n - p \alpha \) with \( \beta \geq 0 \) we have \( \mathcal{H}^{\beta}(\Sigma(f)) = 0 \). As a consequence the Hausdorff dimension of \( \Sigma(f) \) is less or equal to \( n - p \alpha \).

**Proof** It has been shown in Theorem 1 of [16] under the restriction \( 0 < \alpha < \frac{n}{p} \) that \( \mathcal{H}^{\beta}(\Sigma_2(f)) = 0 \). The restriction \( \alpha < \frac{n}{p} \) however was only used to ensure that the case \( \beta < 0 \) cannot occur and \( \mathcal{H}^{\beta} \) is well defined. In our formulation this condition is replaced by \( \beta \geq 0 \). The proof in [16] remains true without any changes. Horihata construct a function \( \phi_\omega \) to which he applies the fundamental lemma of Giusti [14], i.e. \( \mathcal{H}^{\beta}(E_\beta) = 0 \) where

\[
E_\beta := \{ x \in \Omega : \limsup_{\rho \searrow 0} \rho^{-\beta} \int_{B_\rho(x)} |\phi_\omega(y)| dy > 0 \}.
\]

For any \( x_0 \notin E_\beta \) Horihata shows on p. 202 that for \( 0 < R < \tilde{\delta}/2 \) with \( \tilde{\delta} := \text{dist}(x, \partial \Omega) \) holds

\[
|\langle f \rangle_{B_\rho(x_0)} - \langle f \rangle_{B_\rho(x_0)}| \leq c(n, \beta, \varepsilon, \rho) R^{\varepsilon/p}.
\]

But considering p. 203, second line of (19), and p. 204, second line of (24), it can easily be seen that as a byproduct he shows

\[
\int_{B_\rho(x_0)} |f - \langle f \rangle_{B_\rho(x_0)}| dy + \int_{B_\rho(x_0)} |f - \langle f \rangle_{B_\rho(x_0)}| dy \leq c(n, \beta, \varepsilon, \rho) R^{\varepsilon/p}.
\]

The limit \( R \searrow 0 \) directly implies that any \( x_0 \notin E_\beta \) satisfies \( x_0 \notin \Sigma_1(f) \). Therefore, \( \mathcal{H}^{\beta}(E_\beta) = 0 \) gives \( \mathcal{H}^{\beta}(\Sigma_1(f)) = 0 \). This proves the Theorem.

**Remark 15** For \( f \in L^{1}_{\text{loc}}(\mathbb{R}^d) \) let us define

\[
\Sigma_3(f) := \left\{ x \in \Omega : \limsup_{\rho \searrow 0} \int_{B_\rho(x)} |f| dy = \infty \right\}.
\]

Then from

\[
\int_{B_\rho(x)} |f| dy \leq \int_{B_\rho(x)} |f - \langle f \rangle_{B_\rho(x)}| dy + |\langle f \rangle_{B_\rho(x)}|
\]

it follows that \( \Sigma_3(f) \subset \Sigma_1(f) \cup \Sigma_2(f) = \Sigma(f) \).
Remark 16 Please note that it would also be possible to prove Theorem 14 by embeddings from $\mathcal{M}^{2,\alpha}$ the to Bessel potential spaces $L^{2,\alpha}$ with $\varepsilon > 0$ and use the classical capacity estimates for these spaces. The limit $\varepsilon \to 0$ provides an alternative proof of Theorem 14. See [2] for further references.

Remark 17 Note that Theorem 14 and Remark 15 can easily be generalized in the following sense: In the construction of $\Sigma_1$ and $\Sigma_2$ the balls $B_p(x)$ can be replaced by cubes $Q_p(x)$ (sides parallel to the axis). It is even possible to use balls $B_p$ or cubes $Q_p$ (with sides parallel to the axis) which are not centered at $x$ but only contain $x$. This follows easily from the fact that for any $B$ with $x \in B$ the expressions

$$\int_B |f - \langle f \rangle_B| dy \quad \text{and} \quad \int_B |f| dx.$$

with $B \ni x$ can be estimated from above by the same expressions with $B$ replaced by some larger ball $B_p(x)$ centered at $x$.

We will now estimate the singularities of $V(\nabla u)$.

Theorem 18 Let $u$ be a weak solution of system (1.1). Define

$$\alpha := \min \left\{ \frac{1}{1\!\!1 - \alpha_1}, \alpha_1, \frac{1 + \alpha_2}{2} \right\} \leq 1.$$

Then for any $\beta > n - 2\alpha$ with $\beta \geq 0$ holds

$$\mathcal{H}^{(\beta)}(\Sigma(V(\nabla u))) = 0.$$

Especially, the singular set $\Sigma(V(\nabla u))$ has Hausdorff dimension less or equal to $n - 2\alpha$.

Proof Let $Q_j$ be a countable sequence of cubes with $\Omega \subset \bigcup_j Q_j$ and $20Q_j \Subset \Omega$. Then from Theorem 11 we know that $V(\nabla u) \in \mathcal{M}^{2,\alpha}(Q_j)$. Hence, it follows from Theorem 14 that

$$\mathcal{H}^{(\beta)}(\Sigma(V) \cap Q_j) = 0.$$

This immediately implies $\mathcal{H}^{(\beta)}(\Sigma(V)) = 0$ which proves the Theorem.

6 Appendix

For $P_0, P_1 \in \mathbb{R}^{N \times n}$, $\theta \in [0, 1]$ we define $P_\theta := (1 - \theta)P_0 + \theta P_1$. The following fact is standard and can e.g. be found in [1].

Lemma 19 Let $\alpha > -1$ then uniformly in $P_0, P_1 \in \mathbb{R}^{N \times n}$ with $|P_0| + |P_1| > 0$ holds

$$\left( |P_0| + |P_1| \right)^\alpha \sim \int_0^1 |P_\theta|^{\alpha} d\theta. \quad (6.1)$$
Lemma 20 Let $\varphi$ be an $N$-function with $\Delta_2(\{\varphi, \varphi^*\}) < \infty$. Then uniformly for all $P_0, P_1 \in \mathbb{R}^{N \times n}$ with $|P_0| + |P_1| > 0$ holds

$$
\int_0^1 \frac{\varphi'(|P_0|)}{|P_0|} \, d\theta \sim \frac{\varphi'(|P_0| + |P_1|)}{|P_0| + |P_1|},
$$

(6.2)

where the constants only depend on $\Delta_2(\{\varphi, \varphi^*\})$.

Proof From $\varphi(t) \sim t \varphi'(t)$ and the convexity of $\varphi$ we derive

$$
\int_0^1 \frac{\varphi'(|P_0|)}{|P_0|} \, d\theta \geq c \int_0^1 \frac{\varphi(|P_0|)}{|P_0| + |P_1|)^2} \, d\theta \geq c \frac{\varphi(\int_0^1 |P_0| \, d\theta)}{(|P_0| + |P_1|)^2}.
$$

Since by Lemma 19 $\int_0^1 |P_0| \, d\theta \sim |P_0| + |P_1|$ there follows

$$
\int_0^1 \frac{\varphi'(|P_0|)}{|P_0|} \, d\theta \geq c \frac{\varphi(|P_0| + |P_1|)}{|P_0| + |P_1|)2} \geq c \frac{\varphi'(|P_0| + |P_1|)}{|P_0| + |P_1|}.
$$

This proves the first part. Since $\Delta_2(\varphi^*) < \infty$, there exists (as in the proof of Theorem 7) some $\theta \in (0,1)$ and an $N$–function $\rho$ with $\varphi(t) \sim \rho$ and $\Delta_2(\{\rho, \rho^*\}) < \infty$. Note that $\theta$ and $\Delta_2(\{\rho, \rho^*\})$ depend only on $\Delta_2(\{\varphi, \varphi^*\})$. From $\varphi(t) \sim t \varphi'(t)$, $\varphi(t) \sim (\rho(t))^\theta$, and $\rho(t) \sim t \rho'(t)$ we deduce

$$
\int_0^1 \frac{\varphi'(|P_0|)}{|P_0|} \, d\theta \sim \int_0^1 (\rho'(|P_0|)\frac{1}{n} |P_0|^{\frac{1}{\theta}} \, d\theta.
$$

Using the monotonicity of $\rho'$ and Lemma 19 with $\alpha := 1/\theta - 2$ we get

$$
\int_0^1 \varphi'(|P_0|) \, d\theta \leq c \int_0^1 (\rho'(|P_0| + |P_1|))\frac{1}{n} |P_0|^{\frac{1}{\theta}} \, d\theta
$$

$$
\leq c (\rho'(|P_0| + |P_1|))\frac{1}{n} (|P_0| + |P_1|)^{\frac{1}{\theta}} - 2
$$

$$
\sim \varphi'(|P_0| + |P_1|) \frac{1}{|P_0| + |P_1|}.
$$

This proves the lemma.

Lemma 21 Let $\varphi$ be as in Assumption 1. Let $\Phi : \mathbb{R}^{N \times n} \to \mathbb{R}^{\geq 0}$ be given by $\Phi(Q) := \varphi(|Q|)$ and let $\Lambda(Q) := \nabla_{N \times n} \Phi(Q)$. Then $\Lambda(Q) = \varphi(|Q|)\frac{Q}{|Q|}$ for $Q \neq 0$, $\Lambda(0) = 0$, and $\Lambda$ satisfies (2.7).

Proof Note that $\varphi'(0) = 0$, since $\varphi$ is an $N$–function. Observe that for all $Q \in \mathbb{R}^{N \times n}$

$$
(\partial_{jk} \partial_{lm} \Phi)(Q) = \varphi'(|Q|) \left( \frac{\delta_{jk,lm}}{|Q|^2} - \frac{Q_{jk}Q_{lm}}{|Q|^3} \right) + \varphi''(|Q|) \frac{Q_{jk}Q_{lm}}{|Q|^4}.
$$

Especially, with (2.6)

$$
|\partial_{jk} \partial_{lm} \Phi|(Q) \leq c \frac{\varphi'(|Q|)}{|Q|^2} + c \varphi''(|Q|) \leq c \frac{\varphi'(|Q|)}{|Q|}.
$$

(6.3)
Moreover,
\[ A_{jk}(P) - A_{jk}(Q) = (\partial_{jk} \Phi)(P) - (\partial_{jk} \Phi)(Q) = \sum_{lm} \int_0^1 (\partial_{jk} \partial_{lm} \Phi)([Q,P]_s)(P_{lm} - Q_{lm}) ds, \]
where \([Q,P]_s := (1-s) Q + s P\). So by (6.3), Lemma 20, and (2.6)
\[
|A(P) - A(Q)| \leq c \int_0^1 \frac{\varphi'(\|Q,P\|_s)}{\|Q,P\|_s} ds |P - Q| \\
\leq c \frac{\varphi''(\|P\| + \|Q\|)}{|P| + |Q|} |P - Q| \leq c \varphi''(\|P\| + \|Q\|) |P - Q|.
\]

On the other hand due to (2.6) there exists \(\varepsilon > 0\) with \(\varphi'(t)/t > \varepsilon \varphi''(t)\). So by (6.4) for \(G, B \in \mathbb{R}^{n \times n}\) with \(G \neq 0\) holds
\[
\sum_{lm} B_{jk}(\partial_{jk} \partial_{lm} \Phi)(G)B_{lm} \geq \varepsilon \varphi''(\|G\|) |B|^2
\]
This, (6.4), and Lemma 20 imply
\[
\langle A(P) - A(Q), P - Q \rangle \geq \varepsilon \int_0^1 \varphi''(\|Q,P\|_s) |P - Q|^2 ds \\
\geq \varepsilon c \varphi''(\|P\| + \|Q\|) |P - Q|^2.
\]

This proves the lemma.

We will now introduce some auxiliary \(N\)-functions and prove some of their fundamental properties.

**Definition 22** Let \(\varphi, \omega\) be \(N\)-functions with \(\Delta_2\{\varphi, \varphi^*, \omega, \omega^*\} < \infty\). Further assume that \(\omega'(1) = 1\). Then for \(a \geq 0\) we define \(\varphi_{a, \omega}(t) : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}\) by
\[
\varphi_{a, \omega}(t) := \varphi(a + t) \omega\left(\frac{t}{a + t}\right). \quad (6.5)
\]

Further we define \(\varphi_{a, \omega} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}\) by \(\varphi_{a, \omega}(t) := \int_0^t \varphi_{a, \omega}(s) ds\).

By \(\varphi_{a}(t)\) we denote the function \(\varphi_{a, \omega}\) with \(\omega'(1) = t\), i.e.
\[
\varphi_{a}(t) := \frac{t}{a + t}. \quad (6.6)
\]

We remark that the requirement \(\omega'(1) = 1\) is symmetric with respect to \(\omega \leftrightarrow \omega^*\), since \(\omega'(1) = 1\) implies \((\omega^*)'(1) = (\omega')^{-1}(1) = 1\). Thus \(\varphi, \omega\) satisfy the be requirements of Definition 22 if and only if \(\varphi^*, \omega^*\) satisfy the requirements.
Lemma 23 Let $\phi, \omega$ be as in Definition 22. Then for all $a \geq 0$ the function $\phi_{a,0}$ is an $N$-function and $\Delta_2(\{\phi_{a,0}\}_{a \geq 0}) < \infty$, i.e. the family $\phi_{a,0}$ satisfies the $\Delta_2$-condition uniformly in $a \geq 0$.

Proof The assertion is obvious for $a = 0$, since $\phi'_{a,0} = \phi'$. If $a > 0$ then $\phi'(a + t)$ and $\omega'(t/(t + a))$ are strictly increasing, so $\phi'_{a,0}(t)$ is strictly increasing. Moreover, $\phi'_{a,0}(0) = \phi'(a) \omega'(0) = 0$. Thus $\phi_{a,0}$ is an $N$-function.

Due to (2.3) and $\Delta_2(\{\phi, \omega\}) < \infty$ there holds $\phi'(t) \sim \phi'(2t)$ and $\omega'(t) \sim \omega'(2t)$ uniformly in $t \geq 0$. Moreover, for all $a, t \geq 0$ holds $a + 2t \sim a + t$ and $2t/(a + 2t) \sim t/(a + t)$. Thus

$$
\phi'_{a,0}(2t) = \phi'(a + 2t) \omega'(2t/a + 2) \sim \phi'(a + t) \omega'(t/(a + t)) = \phi'_{a,0}(t)
$$

uniformly in $a, t \geq 0$. Again (2.3) implies that $\phi'_{a,0}(2t) \sim \phi_{a,0}(t)$ uniformly in $a, t \geq 0$. This proves the assertion.

Lemma 24 Let $\phi$ satisfy Assumption 1. Then uniformly in $s, t \in \mathbb{R}^n, |s| + |t| > 0$

\begin{equation}
\phi''(|s| + |t|)|s - t| \sim \phi'(|s - t|), \quad \phi''(|s| + |t|)|s - t|^2 \sim \phi'(|s - t|).
\end{equation}

Proof Due to (2.3) and $\Delta_2(\phi) < \infty$ there holds $\phi'(r) \sim \phi'(2r)$ uniformly in $r \geq 0$. Moreover, $|s| + |t| \sim |s| + |s - t|$ uniformly in $s, t \in \mathbb{R}^n$. Thus

$$
\phi''(|s| + |t|) \sim \frac{\phi'(|s| + |t|)}{|s| + |t|} \sim \frac{\phi'(|s| + |s - t|)}{|s| + |s - t|} = \frac{\phi'(|s - t|)}{|s - t|}.
$$

This proves the first inequality in (6.7). The second follows from (2.3).

Lemma 25 Let $\phi$ be as in Assumption 1. Then also $\phi^*$ satisfies the Assumption 1. If we define the $N$-function $\psi$ for $t > 0$ by

$$
\psi'(t) := \sqrt{\phi(t)} t
$$

then $\psi$ and $\psi^*$ satisfy the Assumption 1. Moreover, $\psi''(t) \sim \sqrt{\psi''(t)}$ uniformly in $t > 0$.

Proof From $(\phi^*)'(t) = (\phi')^{-1}(t)$, (2.6), and (2.3) (with $\phi$ replaced by $\phi^*$) we deduce for $t > 0$

$$
(\phi^*)''(t) = \frac{1}{\phi''((\phi^*)'(t))} \sim \frac{((\phi^*)'(t))^2}{\phi^'((\phi^*)'(t))} \sim \frac{((\phi^*)'(t))^2}{\phi^*(t)} \sim \frac{\phi^*(t)}{t^2}.
$$

This proves that $\phi^*$ satisfies Assumption 1. From $\Delta_2(\phi) < \infty$ we deduce $\phi^*(2t) \sim \phi^*(t)$, $\psi^*(2t) \sim \psi^*(t)$, and $\psi(2t) \sim \psi(t)$. Especially, $\Delta_2(\psi^*) < \infty$. Let $K \geq 64$ then with repetitive use of (2.2) and the monotonicity of $\phi^*$ we estimate for all $t \geq 0$

$$
K \psi\left(\frac{2t}{K}\right) \leq 2t \psi\left(\frac{2t}{K}\right) = 2t \sqrt{\phi^\prime\left(\frac{2t}{K}\right)} \frac{2t}{K} \leq 2t \sqrt{\phi^\prime\left(\frac{t}{2}\right)} \frac{2t}{K} = 4 \sqrt{\frac{K}{t}} \psi\left(\frac{t}{2}\right) \leq 8 \psi(t) \leq \psi(t).
$$
Due to (2.4) and (2.5) this is equivalent to $K \psi^*(t/2) \geq \psi^*(t)$ for all $t \geq 0$. This proves $\Delta_2(\psi^*) \leq \infty$. Moreover, for $t > 0$ we deduce from (2.6)

$$\psi''(t) = \frac{1}{2}(t \psi'(t))^{-1/2}(t \psi''(t) + \psi'(t)) \sim \sqrt{\psi''(t)}.$$ 

Overall, we have shown that $\psi$ satisfies Assumption 1. Thus by the first part of the Lemma $\psi''$ also satisfies Assumption 1.

We are now able to prove Lemma 3:

**Proof (Proof of Lemma 3)** Let $A, \varphi, \psi, V$ be as in Lemma 3. Due to (2.7) holds uniformly in $P, Q \in \mathbb{R}^{N \times n}$ and $x \in \Omega$

$$(A(x, P) - A(x, Q)) \cdot (P - Q) \sim |P - Q|^2 \varphi''(|P| + |Q|). \quad (6.8)$$

On the other hand by Lemma 24 holds uniformly in $P, Q \in \mathbb{R}^{N \times n}$

$$|P - Q|^2 \varphi''(|P| + |Q|) \sim \varphi_P(|P - Q|). \quad (6.9)$$

Moreover, by Lemma 25 and Lemma 21 the estimates (2.7) holds with $A$ and $\varphi$ replaced by $V$ and $\psi$. This and Lemma 25 implies

$$|V(P) - V(Q)|^2 \sim ((P - Q)\varphi''(|P| + |Q|))^2 \quad (6.10)$$

$$\sim |P - Q|^2 \varphi''(|P| + |Q|) \quad (6.11)$$

uniformly in $P, Q \in \mathbb{R}^{N \times n}$. The combination of (6.8), (6.9), and (6.10) prove (2.10a), (2.10b), and (2.10c), whereas (2.10d) is just the special case $P = 0$ using $A(0) = V(0) = 0$. This proves the Lemma.

**Lemma 26** Let $\varphi, \omega$ be as in Definition 22. Then

$$(\varphi_{a, \omega})^*(t) \sim (\varphi^*)_{\varphi(a), \omega}(t) \quad (6.12)$$

uniformly in $a, t \geq 0$. Especially, we have uniformly in $t \geq 0$

$$(\varphi_{a})^*(t) \sim (\varphi^*)_{\varphi(a)}(t). \quad (6.13)$$

**Proof** Due to (2.3) and $\Delta_2(\{\varphi, \omega\}) \leq \infty$ there holds $\varphi'(t) \sim \varphi'(2t)$ and $\omega'(t) \sim \omega'(2t)$ uniformly in $t \geq 0$. If $0 \leq t \leq a$ then $a + t \sim a$ and if $a \leq t < \infty$ then $a + t \sim t$. Therefore

$$\varphi'_{a, \omega}(t) = \varphi'(a + t) \omega'(\frac{t}{a + t}) \sim \begin{cases} \varphi'(a) \omega'(\frac{t}{a}) & \text{for } 0 \leq t \leq a, \\ \varphi'(t) \omega'(1) & \text{for } t \geq a \end{cases}$$

uniformly in $a, t \geq 0$. Thus by $\omega'(1) = 1$

$$\varphi'_{a, \omega}(t) \sim \begin{cases} \varphi'(a) \omega'(\frac{t}{a}) & \text{for } 0 \leq t \leq a, \\ \varphi'(t) & \text{for } t \geq a. \end{cases} \quad (6.14)$$
Let $u := \varphi'_{a, \omega}(t)$ then (6.14) implies

\[
((\varphi_{a, \omega})^{'})'(u) = t \sim \begin{cases}
  a(\omega')^{-1}\left(\frac{u}{\varphi(a)}\right) & \text{for } 0 \leq u \leq \varphi_{a, \omega}(a), \\
  \varphi^{-1}(u) & \text{for } u \geq \varphi_{a, \omega}(a),
\end{cases}
\]

uniformly in $a, t \geq 0$. Therefore

\[
((\varphi_{a, \omega})^{'})'(u) = t \sim \begin{cases}
  a(\omega')^{-1}\left(\frac{u}{\varphi(a)}\right) & \text{for } 0 \leq u \leq \varphi_{a, \omega}(a), \\
  \varphi^{-1}(u) & \text{for } u \geq \varphi_{a, \omega}(a).
\end{cases}
\] (6.15)

Because of $\varphi_{a, \omega}(a) = \varphi'(2a)a_{\omega}(\frac{1}{2}) \sim \varphi'(a)$ it is possible in (6.15) to shift the border for $t$ from $\varphi_{a, \omega}(a)$ to $\varphi'(a)$. Especially,

\[
((\varphi_{a, \omega})^{'})'(u) = t \sim \begin{cases}
  a(\omega')^{-1}\left(\frac{u}{\varphi(a)}\right) & \text{for } 0 \leq u \leq \varphi'(a), \\
  \varphi^{-1}(u) & \text{for } u \geq \varphi'(a).
\end{cases}
\] (6.16)

On the other hand we replace in (6.14) $\varphi$ by $\varphi^{*}$, $a$ by $\varphi'(a)$, $\omega$ by $\omega^{*}$, and $t$ by $u$ then

\[
(\varphi^{*})'_{\varphi(a), \omega^{*}}(u) \sim \begin{cases}
  a(\omega')'(\varphi'(a))\left(\omega^{*}\right)\left(\frac{u}{\varphi(a)}\right) & \text{for } 0 \leq u \leq \varphi'(a), \\
  \varphi^{-1}(u) & \text{for } u \geq \varphi'(a).
\end{cases}
\]

Note that $(\omega^{*})'(1) = (\omega')^{-1}(1) = 1$ and $(\varphi^{*})'(\varphi'(a)) = a$, so

\[
(\varphi^{*})'_{\varphi(a), \omega^{*}}(u) \sim \begin{cases}
  a(\omega')'(\varphi'(a))\left(\omega^{*}\right)\left(\frac{u}{\varphi(a)}\right) & \text{for } 0 \leq u \leq \varphi'(a), \\
  \varphi^{-1}(u) & \text{for } u \geq \varphi'(a).
\end{cases}
\] (6.17)

uniformly in $a, t \geq 0$. From (6.16) and (6.17) follows

\[
((\varphi_{a, \omega})^{'})'(u) \sim (\varphi^{*})'_{\varphi(a), \omega^{*}}(u)
\]

uniformly in $a, u \geq 0$. This and (2.3) prove (6.12). Inequality (6.13) follows from (6.12) with the special choice $\omega'(t) = t$.

**Lemma 27** Let $\varphi, \omega$ be as in Definition 22. Then the families $\varphi_{a, \omega}$ and $(\varphi_{a, \omega})^{*}$ satisfy the $\Delta_{2}$-condition uniformly in $a \geq 0$, i.e. it holds $\Delta_{2}(\{\varphi_{a, \omega}\}_{a \geq 0}) < \infty$ and $\Delta_{2}(\{\varphi^{*}_{a, \omega}\}_{a \geq 0}) < \infty$.

**Proof** From Lemma 23 follows $\Delta_{2}(\{\varphi_{a, \omega}\}_{a \geq 0}) < \infty$. By the same lemma we get $\Delta_{2}(\{\varphi^{*}_{a, \omega}\}_{a \geq 0}) < \infty$. Due to Lemma 26 this implies $\Delta_{2}(\{\varphi_{a, \omega}\}_{a \geq 0}) < \infty$. This proves the assertion.

**Lemma 28** Let $\varphi, \omega$ be as in Definition 22. The uniformly in $a, b \in \mathbb{R}^{n}$

\[
\varphi_{\omega, a}(\|a - b\|) \sim \varphi_{\omega, a}(\|a - b\|).
\] (6.18)
Proof The proof is obvious for \( a = b \), so let us assume that \( |a - b| > 0 \). From (2.3) and \(|a| + |a - b| \sim |b| + |a - b|\) we deduce
\[
\frac{\varphi_{|a|,\omega}(|a - b|)}{|a - b|^2} \sim \frac{\varphi'_{|a|,\omega}(|a - b|)}{|a - b|} = \varphi'(|a| + |a - b|) \frac{\omega'}{\omega} \left( \frac{|a - b|}{|a| + |a - b|} \right)
\]
\[
\sim \varphi'(|b| + |a - b|) \frac{\omega'}{\omega} \left( \frac{|a - b|}{|b| + |a - b|} \right) = \frac{\varphi'_{|b|,\omega}(|a - b|)}{|a - b|^2} \]
This proves the assertion.

Lemma 29 Let \( \varphi, \omega \) be as in Definition 22. Then there exists \( c_1 > 0 \) such that for all \( a, b, e \in \mathbb{R}^n \)
\[
\varphi'_{|a|,\omega}(|b - a|) \leq c_1 \varphi'_{|e|,\omega}(|b - e|) + c_1 \varphi'_{|e|,\omega}(|a - e|). \tag{6.19}
\]

Proof If \(|b - e| \leq |a - e|\) then \(|a - b| \leq 2|a - e|\) and
\[
\varphi'_{|a|,\omega}(|b - a|) \leq \varphi'_{|a|,\omega}(2|a - e|)
\]
\[
\sim \varphi'_{|a|,\omega}(|a - e|) \quad \text{by Lemma 23} \tag{6.20}
\]
\[
\sim \varphi'_{|a|,\omega}(|a - e|) \quad \text{by (6.18)}
\]
This proves the assertion in the case \(|b - e| \leq |a - e|\). Assume now that \(|a - e| \leq |b - e|\). From (2.3) and (6.18) we know \( \varphi'_{|a|,\omega}(|a - b|) \sim \varphi'_{|b|,\omega}(|a - b|) \). The rest follows from (6.20) with \( a \) and \( b \) interchanged.

Lemma 30 Let \( \varphi, \omega \) be as in Definition 22. Then uniformly in \( \lambda \in [0, 1] \) and \( a \geq 0 \) holds
\[
\varphi_{\omega,\omega}(\lambda) a \sim \omega(\lambda) \varphi(a). \tag{6.21}
\]
Especially,
\[
\varphi_0(\lambda a) \sim \lambda^2 \varphi(a), \tag{6.22}
\]
\[
(\varphi_0)^*(\lambda \varphi(a)) \sim \lambda^2 \varphi(a). \tag{6.23}
\]

Proof Because of (2.3) and (6.14) holds
\[
\varphi_{\omega,\omega}(\lambda) a \sim \lambda a \varphi_{\omega,\omega}(\lambda) a \sim \lambda a \varphi'(a) \omega'(\lambda) \sim \varphi(a) \omega(\lambda).
\]
This proves (6.21) while (6.22) is a special case of (6.21) with \( \omega'(t) = t \). Moreover, (6.13), (6.22), and (2.3) imply
\[
(\varphi_0)^*(\lambda \varphi(a)) \sim (\varphi^0)\varphi(a) (\lambda \varphi(a)) \sim \lambda^2 \varphi^*(\varphi(a)) \sim \lambda^2 \varphi(a).
\]
So, (6.23) is proven.
Lemma 31 Let \( \varphi \) be an N-function with \( \Delta_2(\{\varphi, \varphi^{*}\}) < \infty \). Then there exist \( \varepsilon > 0 \), \( c_2 > 0 \) which only depend on \( \Delta_2(\{\varphi, \varphi^{*}\}) \) such that for all \( t \geq 0 \) and all \( \lambda \in [0,1] \)

\[
\varphi(\lambda t) \leq c_2 \lambda^{1+\varepsilon} \varphi(t). \tag{6.24}
\]

In particular, there exists \( \varepsilon > 0 \) and \( c_2 > 0 \) such that

\[
\varphi_a(\lambda t) \leq c_2 \lambda^{1+\varepsilon} \varphi_a(t) \tag{6.25}
\]

uniformly in \( a, t \geq 0 \) and \( \lambda \in [0,1] \).

Proof Since \( \Delta_2(\{\varphi, \varphi^{*}\}) < \infty \) there exists, as in Theorem 7, an N-function \( \rho \) and \( \theta \in (0,1) \) with \( \varphi^\theta \sim \rho \). This implies uniformly in \( t \geq 0 \) and \( \lambda \in [0,1] \)

\[
\varphi(\lambda t) \sim \left( \rho(\lambda t) \right)^\frac{1}{\theta} \leq \left( \lambda \rho(t) \right)^\frac{1}{\theta} \sim \lambda^{\frac{1}{\theta}} \varphi(t),
\]

where we have used the convexity of \( \rho \) and \( \rho(0) = 0 \). Inequality (6.24) follows with \( \varepsilon := \frac{1}{\theta} - 1 \). Now, (6.25) follows from Lemma 27.

Lemma 32 (Young type inequality) Let \( \varphi \) be an N-function which fulfills \( \Delta_2(\{\varphi, \varphi^{*}\}) < \infty \). Then for all \( \delta > 0 \) there exists \( c_\delta \) which only depends on \( \delta \) and \( \Delta_2(\{\varphi, \varphi^{*}\}) \) such that for all \( t, u \geq 0 \)

\[
t u \leq \delta \varphi(t) + c_\delta \varphi^*(u), \tag{6.26a}
\]

\[
t \varphi'(u) + \varphi'(t) u \leq \delta \varphi(t) + c_\delta \varphi(u). \tag{6.26b}
\]

Let \( \varphi, \omega \) be as in Definition 22. Then for all \( \delta > 0 \) there exists \( c_\delta \) such that for all \( t, u, a \geq 0 \)

\[
t \varphi_a(t) \leq \delta \varphi_a(t) + c_\delta \varphi_a^*(u), \tag{6.27}
\]

\[
t \varphi_a'(u) + \varphi_a'(t) u \leq \delta \varphi_a(t) + c_\delta \varphi_a(u). \tag{6.28}
\]

Proof Inequality (6.26a) is well known, see (2.1). Now (6.26b) follows from (6.26a) and (2.3). Because of Lemma 27 we can apply (6.26a) and (6.26b) to the family \( \{\varphi_a\}_{a \geq 0} \). This proves (6.27) and (6.28).

Remark 33 Note that Lemma 32 together with Lemma 3 generalize many known estimates. One example are the quasi-norms estimates of Barrett and Liu in [3].

Lemma 34 Let \( \varphi, \sigma, \kappa, \omega \) be N-function with \( \sigma'(1) = \kappa'(1) = \omega'(1) = 1 \) and \( \Delta_2(\{\varphi, \varphi^{*}, \sigma, \sigma^{*}, \kappa, \kappa^{*}, \omega, \omega^{*}\}) < \infty \). Moreover, let

\[
\kappa(t) \sim \sigma^*(\omega'(t)) \tag{6.29}
\]

uniformly in \( t \geq 0 \). Then uniformly in \( a, t \geq 0 \)

\[
(\varphi_a, \sigma)^* \left( \varphi_a^*, \omega(t) \right) \sim \varphi_a, \kappa(t). \tag{6.30}
\]

Moreover, for every \( \delta > 0 \) there exists \( c_\delta > 0 \) such that uniformly in \( a, t, u \geq 0 \)

\[
\varphi_a'(t) u \leq \delta \varphi_a, \kappa(t) + c_\delta \varphi_a, \sigma(u). \tag{6.31}
\]
Proof Let us remark that if $\sigma = \kappa = \omega$ then (6.29) and (6.30) follow immediately from (2.3).

From (6.14) and (6.17) we deduce

$$
\varphi'_{a,\omega}(t) \sim \begin{cases} 
\varphi'(a) \omega'\left(\frac{t}{a}\right) & \text{for } 0 \leq t \leq a, \\
\varphi'(t) & \text{for } t \geq a.
\end{cases} \quad (6.32)
$$

$$
((\varphi_{a,\sigma})')'(u) \sim (\varphi')'_{\varphi'(a),\sigma'}(u) \sim \begin{cases} 
a(\sigma')'\left(\frac{u}{\varphi'(a)}\right) & \text{for } 0 \leq u \leq \varphi'(a), \\
(\varphi')'(u) & \text{for } u \geq \varphi'(a).
\end{cases} \quad (6.33)
$$

Because of $\varphi'_{a,\omega}(a) = \varphi'(2a) \sigma(\frac{1}{2}) \sim \varphi'(a)$ it is possible in (6.15) to shift the border for $u$ from $\varphi'(a)$ to $\varphi'_{a,\omega}(a)$, i.e.

$$
((\varphi_{a,\sigma})')'(u) \sim \begin{cases} 
a(\sigma')'\left(\frac{u}{\varphi'(a)}\right) & \text{for } 0 \leq u \leq \varphi'_{a,\omega}(a), \\
(\varphi')'(u) & \text{for } u \geq \varphi'_{a,\omega}(a).
\end{cases}
$$

Repeatedly use of (2.3) implies

$$
(\varphi_{a,\sigma})^* (\varphi'(a) \sigma'(t)) \sim \begin{cases} 
a(a(\sigma')'(\frac{u}{\varphi'(a)})) & \text{for } 0 \leq u \leq \varphi'_{a,\omega}(a), \\
\varphi'(u) & \text{for } u \geq \varphi'_{a,\omega}(a).
\end{cases} \quad (6.34)
$$

Now the composition of (6.32) and (6.34) gives

$$
(\varphi_{a,\sigma})^* (\varphi'_{a,\omega}(t)) \sim \begin{cases} 
\varphi(a) \sigma'(\frac{\omega'(t)}{a}) & \text{for } 0 \leq t \leq a, \\
\varphi'(\varphi'(t)) & \text{for } t \geq a
\end{cases}
$$

$$
\sim \begin{cases} 
\varphi(a) \kappa(\frac{t}{a}) & \text{for } 0 \leq t \leq a, \\
\varphi(t) & \text{for } t \geq a
\end{cases}
\sim \varphi_{a,\kappa}(t) \quad \text{by (6.14) and (2.3)}.
$$

Now (6.14) concludes

$$
(\varphi_{a,\sigma})^* (\varphi'_{a,\omega}(t)) \sim \varphi_{a,\kappa}(t)
$$

This proves (6.30). Now (6.31) is a direct implication of (6.30) and Young’s inequality (6.28).

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References