COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XIV. 415–426 (1961)

K. O. Friedrichs anniversary issue

On Functions of Bounded Mean Oscillation*

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§ 1. We prove an inequality (Lemmas 1.1′) which has been applied by one of the authors and by J. Moser in their papers in this issue. The inequality expresses that a function, which in every subcube C of a cube C_0 can be approximated in the L^1 mean by a constant a_C with an error independent of C, differs then also in the L^p mean from a_C in C by an error of the same order of magnitude. More precisely, the measure of the set of points in C, where the function differs from a_C by more than an amount σ decreases exponentially as σ increases.

In Section 2 we apply Lemma 1′ to derive a result of Weiss and Zygmund [3], and in Section 3 we present an extension of Lemma 1′.

LEMMA 1. *Let $u(x)$ be an integrable function defined in a finite cube C_0 in n-dimensional space; $x = (x_1, \cdots, x_n)$. Assume that there is a constant K such that for every parallel subcube C, and some constant a_C, the inequality*

$$\frac{1}{m(C)}\int_C |u-a_C|dx \leqq K \tag{1}$$

holds. Here dx denotes element of volume and $m(C)$ is the Lebesgue measure of C. Then, if $\mu(\sigma)$ is the measure of the set of points where $|u-a_{C_0}| > \sigma$, we have

$$\mu(\sigma) \leqq Be^{-b\sigma/K} m(C_0) \quad \textit{for} \quad \sigma > 0, \tag{2}$$

where B, b are constants depending only on n.

Since for every continuously differentiable function $f(s)$, vanishing at the origin,

$$\int_{C_0} f(|u-a_{C_0}|)dx = \int_0^\infty \mu(s)df(s),$$

inequality (2) implies that u belongs to L^p for every finite $p \geqq 1$, and, in fact for $b' < K^{-1}b$ the function $e^{b'|u-a_{C_0}|}$ is integrable and

$$\int_{C_0} e^{b'|u-a_{C_0}|}dx \leqq \left(1 + \frac{Bb'}{K^{-1}b-b'}\right) m(C_0). \tag{3}$$

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A function satisfying (1) for every subcube C of C_0, for come constant a_C, will be said to have "mean oscillation $\leqq K$ in C_0". Taking for a_C the average of u in C we always have

$$\left(\frac{1}{m(C)}\int_C |u-a_C|dx\right)^2 \leqq \frac{1}{m(C)}\int_C |u(x)-a_C|^2 dx$$
$$= \tfrac{1}{2}(m(C))^{-2}\int_C dx\int_C dy|u(x)-u(y)|^2.$$

In particular u has mean oscillation $\leqq K$, if u is bounded and its oscillation $|u(x)-u(y)|$ does not exceed the value $\sqrt{2}K$ in C_0.

Boundedness of u is not necessary for boundedness of its mean oscillation. Let indeed $u(x)$ be any integrable function in C_0 with the property that we can associate with every subcube C a value a_C such that the subset S_σ of C, where

$$|u-a_C| \geqq \sigma,$$

has measure

$$\mu(\sigma) \leqq Be^{-b\sigma}m(C) \quad \text{for} \quad \sigma > 0.$$

Then

$$\int_C |u-a_C|dx = \int_0^\infty \mu(\sigma)d\sigma \leqq \frac{B}{b}\, m(C)$$

so that the mean oscillation of u does not exceed B/b. Take now for u the function $\log|x-y|$, where y is fixed. Let C be any cube of side h, and let ξ and η be points of C for which

$$|\xi-y| = \operatorname*{Max}_{x\in C} |x-y|, \qquad |\eta-y| = \operatorname*{Min}_{x\in C} |x-y|.$$

Take $a_C = \log|\xi-y|$. Then for x in C

$$|u(x)-a_C| = \log\frac{|\xi-y|}{|x-y|}.$$

S_σ is the subset of C lying in the sphere

$$|x-y| \leqq |\xi-y|e^{-\sigma}.$$

If S_σ is not empty, it must contain η, so that

$$|\xi-y|e^{-\sigma} \geqq |\eta-y| \geqq |\xi-y|-|\xi-\eta| \geqq |\xi-y|-\sqrt{n}\,h.$$

Thus

$$|\xi-y| \leqq \frac{\sqrt{n}\,h}{1-e^{-\sigma}}.$$

It follows that S_σ is contained in the sphere

$$|x-y| \leqq \frac{\sqrt{n}\, h}{e^\sigma - 1}$$

and that its measure $\mu(\sigma)$ does not exceed

$$\left(\frac{\sqrt{n}\omega_n}{e^\sigma - 1}\right)^n m(C),$$

where the volume of the unit sphere in n-space is denoted by $(\omega_n)^n$. Since also $\mu(\sigma) \leqq m(C)$ for all $\sigma \geqq 0$, we find that

$$\mu(\sigma) \leqq (1+\sqrt{n}\omega_n)^n e^{-n\sigma} m(C) = Be^{-b\sigma} m(C) \quad \text{for} \quad \sigma > 0,$$

where B and b do not depend on C. This proves that $\log|x-y|$ is of bounded mean oscillation in every cube C_0. The same holds then for any function $u(x)$ of the form

$$u(x) = \int \zeta(y) \log|x-y| dy \quad \text{with} \quad \int |\zeta(y)| dy < \infty.$$

Lemma 1 will be derived from

LEMMA 1′. *Let $u(x)$ be integrable in a cube C_0 and assume that there is a constant κ such that for every parallel subcube C we have*

$$\frac{1}{m(C)} \int_C |u - u_C| dx \leqq \kappa, \tag{1$'$}$$

where u_C is the mean value of u in C. Then if S_σ is the set of points where $|u-u_{C_0}| > \sigma$, its measure $m(S_\sigma)$ satisfies

$$m(S_\sigma) \leqq \frac{A}{\kappa} \int_{C_0} |u - u_{C_0}| dx \cdot e^{-\alpha\sigma\kappa^{-1}} \quad \textit{for} \quad \frac{\sigma}{\kappa} \geqq a. \tag{2$'$}$$

Since $m(S_\sigma) \leqq m(C_0)$ it follows that

$$m(S_\sigma) \leqq e^{\alpha a} e^{-\alpha\sigma\kappa^{-1}} m(C_0) \quad \textit{for} \quad \sigma > 0. \tag{2$''$}$$

Here $A \leqq 1$, α, a are positive numbers depending only on the dimension n.

By a standard type of argument we can derive, as consequences of (2)′, the following inequalities: for $0 < \beta < \alpha$,

$$\int_{C_0} e^{\beta\kappa^{-1}|u-u_{C_0}|} dx \leqq \left(\frac{\alpha}{\alpha-\beta} + e^{\beta a}\right) m(C_0), \tag{3$'$}$$

$$
\begin{aligned}
(3)'' \qquad \int_{C_0} (e^{\beta\kappa^{-1}|u-u_{C_0}|}-1)dx &\leqq \left\{\frac{e^{\beta\alpha}-1}{\beta a}+\frac{A}{\kappa}\frac{\alpha}{\alpha-\beta}e^{a(\beta-\alpha)}\right\}\int_{C_0}|u-u_{C_0}|dx \\
&= \tilde{A}\int_{C_0}|u-u_{C_0}|dx \\
&\leqq 2\tilde{A}\int_{C_0}|u|dx.
\end{aligned}
$$

The inequality (3)'' is of interest since, in case $u(x)$ is integrable in an *infinite* cube $\tilde{C}_0$ and satisfies (1)' in every finite subcube, we can conclude from (3)'' that

$$
(3)''' \qquad \int_{\tilde{C}_0} (e^{\beta\kappa^{-1}|u|}-1)dx \leqq 2\tilde{A}\int_{\tilde{C}_0}|u|dx.
$$

If one wishes to prove (2)'' directly, without proving (2)', then the proof given below can be simplified slightly.

Lemma 1 follows easily from Lemma 1'; for (1) implies that $|u_C-a_C|\leqq K$ so that

$$
\frac{1}{m(C)}\int_C |u-u_C|dx \leqq 2K.
$$

By Lemma 1', (2)'' holds, with $\kappa = 2K$, and (2) then follows easily.

The proof of Lemma 1' is based on a decomposition of integrable functions which, in one dimension, is due to F. Riesz, and which has been used extensively by Calderon and Zygmund [1] and Hörmander [2]. For completeness we include the proof of this decomposition, in a form suitable for application to Lemma 1'.

DECOMPOSITION. Let u be an integrable function defined in a cube C_0 and let s be a positive number such that

$$
(4) \qquad s \geqq \frac{1}{m(C_0)}\int_{C_0}|u|dx.
$$

There exists a denumerable number of open disjoint cubes I_k in C_0 such that

i) $|u| \leqq s$ a.e. in $C_0-\bigcup_k I_k$,

ii) the average value u_k of u in I_k is bounded in absolute value by $2^n s$,

iii) $\sum_k m(I_k) \leqq s^{-1}\int_{C_0}|u|dx$.

Proof: Divide C_0 (by halving each edge) into 2^n equal cubes and let $I_{11}, I_{12}, \cdots$ be those open cubes over which the average value of $|u|$ is $\geqq s$. Then

$$
sm(I_{1k}) \leqq \int_{I_{1k}}|u|dx \leqq 2^n s\, m(I_{1k})
$$

by (4). Next subdivide each remaining cube, over which the average of $|u|$ is $< s$, into 2^n equal cubes, and denote by $I_{21}, I_{22}, \ldots$ those cubes thus obtained over which the average of $|u|$ is $\geqq s$. Again subdivide the remaining cubes, etc. In this way we obtain a sequence of cubes I_{ik}, which we rename I_k, such that

$$sm(I_k) \leqq \int_{I_k} |u| dx < 2^n sm(I_k).$$

Clearly property ii) is satisfied. Furthermore, summing the left inequality over k we obtain iii). We observe finally that a point of C_0 which does not belong to any of the I_k belongs to arbitrarily small cubes over which the average of $|u|$ is $< s$. Hence $|u| \leqq s$ a.e. outside all the I_k, verifying i).

Proof of Lemma 1′: We may assume without loss of generality that $u_{C_0} = 0$ and that $\kappa = 1$, by replacing u by $(u - u_{C_0})/\kappa$.

Denote by $F(\sigma)$ the smallest number, depending only on σ and n (and independent of the particular function u or cube C_0) such that

$$m(S_\sigma) \leqq F(\sigma) \int_{C_0} |u| dx;$$

obviously $F(\sigma) \leqq 1/\sigma$. We now prove that, for $\sigma \geqq 2^n$,

$$F(\sigma) \leqq \frac{1}{s} F(\sigma - 2^n s) \quad \text{for} \quad 2^{-n}\sigma \geqq s \geqq 1. \tag{5}$$

To this end we apply the above decomposition to the function u in C_0 with

$$2^{-n}\sigma \geqq s \geqq 1 \geqq \frac{1}{m(C_0)} \int_{C_0} |u| dx,$$

the last inequality following from (1)′. Because of i) we see that if $|u(x)| > \sigma$, then x belongs to one of the I_k (except for a set of measure zero). Hence, since the average u_k of u in I_k is bounded by $2^n s$ in absolute value, we see that

$$m(S_\sigma) = m\{x \big| |u(x)| > \sigma\} \leqq \sum_k m\{x \big| |u(x) - u_k| > \sigma - 2^n s \quad \text{in} \quad I_k\}.$$

Now in the cube I_k the function $u - u_k$ satisfies the hypotheses of Lemma 1′, in particular it satisfies (1)′ for every cube in I_k. Hence, using the definition of $F(\sigma)$, we have

$$\begin{aligned} m\{x \big| |u(x) - u_k| > \sigma - 2^n s \text{ in } I_k\} &\leqq F(\sigma - 2^n s) \int_{I_k} |u - u_k| dx \\ &\leqq F(\sigma - 2^n s) m(I_k). \end{aligned}$$

Thus we find

$$m(S_\sigma) \leqq F(\sigma-2^n s) \sum_k m(I_k) \leqq \frac{1}{s} F(\sigma-2^n s) \int_{C_0} |u| dx$$

by iii), proving (5).

Setting $s = e$ in (5) we see that if $F(\sigma) \leqq Ae^{-\alpha\sigma}$, $\alpha = 1/(2^n e)$ for some σ, then

$$F(\sigma+2^n e) \leqq \frac{1}{e} Ae^{-\alpha\sigma} = Ae^{-\alpha(\sigma+2^n e)}.$$

From this it follows that if on some interval of length $2^n e$ the inequality $F(\sigma) \leqq Ae^{-\alpha\sigma}$ holds, then it holds for all larger σ. But a calculation shows that

$$F(\sigma) \leqq \frac{1}{\sigma} \leqq \frac{12}{10} 2^{-n} e^{-\alpha\sigma} \quad \text{for} \quad \frac{2^n e}{e-1} \leqq \sigma \leqq \frac{2^n e}{e-1} + 2^n e.$$

(This interval is the one of length $2^n e$ on which the maximum of $e^{\alpha\sigma}/\sigma$ is as small as possible.) Thus we conclude that

$$m(S_\sigma) \leqq \frac{12}{10} 2^{-n} e^{-\alpha\sigma} \int_{C_0} |u| dx \quad \text{for} \quad \alpha\sigma \geqq \frac{1}{e-1}, \quad \alpha = \frac{1}{2^n e},$$

that is, we have proved (2)′ with

$$A = \frac{12}{10} 2^{-n}, \quad \alpha = \frac{1}{2^n e}, \quad a = \frac{2^n e}{e-1}. \tag{6}$$

We have made no attempt here to obtain the best constants. The exponent α can be considerably improved, i.e. increased, by using the hypothesis (1)′ again to sharpen the estimate $|u_k| \leqq 2^n s$ that was provided by ii). We mention only that we have proved (2)′ with a constant α which for large n behaves like $(1/e \log 2)$ $(\log n/n)$.

§ 2. A recent paper by M. Weiss and A. Zygmund [3] contains the following

THEOREM. *If $F(x)$ is periodic and for some $\beta > \frac{1}{2}$ satisfies*

$$F(x+h)+F(x-h)-2F(x) = O\left(\frac{h}{|\log h|^\beta}\right) \tag{7}$$

uniformly in x, then F is the indefinite integral of an f belonging to every L_p.

They also give an example showing that the result does not hold for $\beta = \frac{1}{2}$.

The proof of the theorem in [3] is rather short but it relies on a theorem of Littlewood and Paley, and it seems of interest to us to show how it may be derived from our Lemma 1′: we prove

LEMMA 2. *Let $u(x)$ be an integrable function defined in a finite cube C_0 in n-space. Assume that there is a constant K and a constant $\beta > \frac{1}{2}$ such that if C_1 and C_2, are any two equal subcubes having a full $(n-1)$-dimensional face in common, then*

$$|u_{C_1} - u_{C_2}| \leqq \frac{K}{1+|\log h|^{\beta}}; \tag{8}$$

here u_{C_1}, u_{C_2} are the mean values of u in the cubes C_1 and C_2, and h is the common side length. Then u satisfies the conditions of Lemma 1′ with some constant κ depending on K, β and n so that, consequently, u satisfies (2)″ and (3)′.

The preceding theorem follows easily from this lemma. By convolution of F with a smooth peaked kernel we may suppose that F is infinitely differentiable. It suffices merely to estimate the L_p norm of the derivative f of F. Hypothesis (7) asserts simply that f satisfies (8) for $n = 1$. Applying Lemma 2 we obtain from (2)″ or (3)″ an estimate for the L_p norm of f depending only on K and β, proving the theorem. From (3)″ we find, furthermore, that $e^{\alpha'|f|}$ is integrable for some $\alpha' > 0$.

Proof of Lemma 2: Consider a subcube C, of side length k subdivided into 2^{nN} equal cubes C_r, $r = 1, \cdots, 2^{nN}$, obtained by dividing each edge into 2^N equal parts, and let u_r denote the mean value of u in C_r. Then

$$\frac{1}{m(C)}\int_C |u - u_C| dx = \lim_{N\to\infty} 2^{-nN} \sum_r |u_r - u_C|.$$

Thus to prove (1)′ it suffices to show that $2^{-nN}\sum |u_r - u_C| \leqq \kappa$, with κ depending only on K, β and n.

By Schwarz inequality,

$$2^{-nN}\sum |u_r - u_C| \leqq [2^{-nN}\sum |u_r - u_C|^2]^{1/2} = a_N{}^{1/2}.$$

We shall prove that the a_N are uniformly bounded by showing that

$$a_{N+1} \leqq a_N + \left(\frac{nK}{1+|\log h|^{\beta}}\right)^2, \qquad h = \frac{k}{2^{N+1}}. \tag{9}$$

Since $a_0 = 0$, it follows that

$$a_{N+1} \leqq n^2K^2 \sum_{j=1}^{\infty} (1+|j \log 2 - \log k|^{\beta})^{-2} \leqq \kappa^2$$

for some constant κ independent of k, convergence being guaranteed by the fact that $\beta > \frac{1}{2}$.

Thus to complete the proof we shall establish (9). We observe first that $2^{-nN}\sum_r u_r = u_C$ so that using the general identity

$$k \sum_{1}^{k} b_r^2 = (\sum b_r)^2 + \tfrac{1}{2} \sum_{r,s} (b_r - b_s)^2$$

for real b_r, we find

$$2^{2nN} a_N = 2^{nN} \sum |u_r - u_C|^2 = \tfrac{1}{2} \sum_{r,s} |u_r - u_s|^2. \tag{10}$$

Now, on the next subdivision of C into $2^{n(N+1)}$ cubes each C_r is divided into 2^n equal cubes C_{ri}, $i = 1, \cdots, 2^n$, of side length $h = k/2^{N+1}$. If u_{ri} is the mean value of u in C_{ri} we have

$$u_r = 2^{-n} \sum_i u_{ri}. \tag{11}$$

Furthermore, since any two C_{ri}, C_{rj} can be connected by a chain of at most $n+1$ cubes each having a full face in common with the succeeding one, we find from (8) that

$$|u_{ri} - u_{rj}| \leqq \frac{nK}{1 + |\log h|^\beta} = M,$$

where M is so defined. This together with (11) implies

$$|u_{ri} - u_r| \leqq M.$$

According to formula (10)

$$\begin{aligned} 2.2^{2n(N+1)} a_{N+1} &= \sum_{\substack{r,s \leqq 2^{nN} \\ i,j \leqq 2^n}} |u_{ri} - u_{sj}|^2 \\ &= \sum [(u_{ri}^2 + u_{sj}^2) - 2u_{ri}u_{sj}] \\ &= \sum [(u_{ri} - u_r)^2 + (u_{sj} - u_s)^2 + 2u_{ri}u_r \\ &\quad + 2u_{sj}u_s - u_r^2 - u_s^2 - 2u_{ri}u_{sj}] \\ &= \sum_{\substack{r,s \\ i,j}} [(u_{ri} - u_r)^2 + (u_{sj} - u_s)^2] \\ &\quad + \sum_{r,s} [2.2^{2n}(u_r^2 + u_s^2) - 2^{2n}(u_r^2 + u_s^2) - 2.2^{2n} u_r u_s], \end{aligned}$$

by (11),

$$\begin{aligned} &\leqq 2M^2 2^{2nN+2n} + 2^{2n} \sum_{r,s} (u_r - u_s)^2 \\ &= 2.2^{2n(N+1)} M^2 + 2.2^{2n(N+1)} a_N, \end{aligned}$$

by (10), or

$$a_{N+1} \leqq a_N + M^2.$$

This is the desired inequality (9) and the proof is complete.

§ 3. In this section we present briefly a generalization of Lemma 1′.

LEMMA 3. *Let u be integrable in a finite cube C_0 and consider a subdivision of C_0 into a denumerable number of cubes C_i, no two having a common interior point. Assume that for fixed p, $1 < p < \infty$, the expression*

$$\left\{\sum_i m(C_i)^{1-p}\left|\int_{C_i}|u-u_{C_i}|dx\right|^p\right\}^{1/p}$$

is finite. Denote by K_u the lim sup of such expressions for all possible subdivisions of C_0 of this kind: in general $K_u = \infty$. If $K_u < \infty$, the measure $m(S_\sigma)$ of the set S_σ, where $|u-u_{C_0}| > \sigma$, satisfies

$$m(S_\sigma) \leqq A\left|\frac{K_u}{\sigma}\right|^p \quad \text{for} \quad \sigma > 0,$$

for some constant A depending only on n and p.

The result implies that the function u belongs to $L^{p'}$ for every $p' < p$.

For $p = \infty$ the hypothesis of Lemma 3 agrees with that of Lemma 1′.

Proof: We shall not attempt to obtain the best constants. Let $q = p/(p-1)$ be the conjugate exponent to p. We may assume that $u_{C_0} = 0$. Using induction with respect to the integer j we shall prove that if

$$s = \frac{2^{-n}\sigma}{p(q^j-1)+1} \geqq \frac{K_u}{m(C_0)^{1/p}}, \tag{12}$$

then

$$m(S_\sigma) \leqq 2^{-n}q^{1/q+2/q^2+\cdots+j/q^j}\left|\frac{2^n p(1-q^{-1-j})K_u}{\sigma}\right|^{p(1-1/q^{j+1})}\left(\frac{1}{K_u}\int_{C_0}|u|dx\right)^{1/q^j}. \tag{13}$$

Since

$$m(S_\sigma) \leqq \frac{1}{\sigma}\int_{C_0}|u|dx,$$

(13) holds for $j = 0$. Suppose then it is true for $j-1$, we wish to prove it for j.

Since

$$\frac{1}{m(C_0)}\int_{C_0}|u|dx \leqq \frac{K_u}{m(C_0)^{1/p}}, \tag{14}$$

we may apply the decomposition of § 1 to u, with s equal to its value in (12). Let u_k denote the mean value of u in I_k, and set $v_k = u-u_k$ in I_k. From the definition of K_u we may assert that

$$\sum_k K_{v_k}^p \leqq K_u^p. \tag{15}$$

Setting $a_k = \int_{I_k} |v_k| dx$ we note further (as in (14)) that

$$m(I_k)^{1-p} a_k^p \leqq K_{v_k}^p, \tag{16}$$

so that by Hölder's inequality

$$\begin{aligned} \sum a_k &\leqq \left(\sum m(I_k)^{1-p} a_k^p\right)^{1/p} \left|\sum m(I_k)\right|^{1/q} \\ &\leqq \left(\sum K_{v_k}^p\right)^{1/p} \left|\sum m(I_k)\right|^{1/q}, \end{aligned}$$

or

$$\sum a_k \leqq K_u \left| s^{-1} \int_{C_0} |u| dx \right|^{1/q} \tag{17}$$

by (15) and iii).

As in the derivation of (5), we have

$$m(S_\sigma) \leqq \sum_k m\{x \in I_k \,\big|\, |v_k| > \sigma - 2^n s\}.$$

Applying the induction hypothesis (13), for $j-1$, to the functions v_k in I_k we find

$$\begin{aligned} m(S_\sigma) &\leqq \left[2^{-n} q^{1/q + \cdots + (j-1)/q^{j-1}} \left| \frac{2^n p (1-q^{-j})}{\sigma - 2^n s} \right|^{p(1-1/q^j)} \right] \\ &\quad \cdot \sum_k K_{v_k}^{p(1-1/q^j)} \left(\frac{1}{K_{v_k}} \int_{I_k} |v_k| dx \right)^{1/q^{j-1}} \\ &= [\ ] \sum_k K_{v_k}^{p(1-q^{1-j})} a_k^{1/q^{j-1}} \\ &\leqq [\ ] \left(\sum a_k\right)^{1/q^{j-1}} \left(\sum K_{v_k}^p\right)^{1-q^{1-j}}, \end{aligned}$$

by Hölder's inequality,

$$\leqq [\ ] K_u^{1/q^{j-1}} \left| s^{-1} \int_{C_0} |u| dx \right|^{1/q^j} K_u^{p - pq^{1-j}}$$

by (17) and (15), so that

$$m(S_\sigma) \leqq \frac{[\ ]}{s^{1/q^j}} K_u^{p(1-1/q^{j+1})} \left(\frac{1}{K_u} \int_{C_0} |u| dx \right)^{1/q^j}.$$

A slightly tedious calculation shows that this inequality is identical with the desired result (13).

Having established (13) we may now express it in a more convenient form: if (12) holds, then, in virtue of (14), there is a constant k depending only on n and p such that

$$m(S_\sigma) \leqq k \left(\frac{K_u}{\sigma} \right)^{p(1-1/q^{j+1})} m(C_0)^{1/q^{j+1}}$$

or

$$m(S_\sigma) \leqq k\left(\frac{K_u}{\sigma}\right)^p \cdot \left|\frac{\sigma m(C_0)^{1/p}}{K_u}\right|^{p/q^{j+1}}.$$

If now $2^{-n}\sigma \geqq K_u m(C_0)^{-1/p}$ and we choose the largest integer $j \geqq 0$ so that (12) is satisfied, we have the opposite inequality for $j+1$:

$$\frac{\sigma m(C_0)^{1/p}}{K_u} \leqq 2^n\left(p(q^{j+1}-1)+1\right) \leqq 2^n p q^{j+1}.$$

Inserting into the previous inequality we find

$$\begin{aligned} m(S_\sigma) &\leqq k\left(\frac{K_u}{\sigma}\right)^p |2^n p q^{j+1}|^{p/q^{j+1}} \\ &\leqq A\left(\frac{K_u}{\sigma}\right)^p \quad \text{for} \quad \sigma \geqq 2^n K_u m(C_0)^{-1/p}, \end{aligned}$$

for some constant A depending only on n and p. Since $m(S_\sigma) \leqq m(C_0)$, the same inequality holds for all $\sigma > 0$, with some other constant A, and the proof of the lemma is complete.

§ 4. Inequality (2)″ in Lemma 1′ can be replaced by the more general inequality

$$(2)''' \qquad m(S_{2^{n+2}\sigma}) \leqq A e^{-B\sigma\kappa^{-1}} m(S_\sigma) \quad \text{for} \quad \sigma > 0$$

with A, B depending only on n.

Proof: Let $\kappa = 1$, $u_{C_0} = 0$. For a fixed positive s the cubes I_k shall be defined as in the proof of Lemma 1′. Put

$$\mu_k(\sigma) = m(x|\ |u(x)| > \sigma \text{ in } I_k).$$

By definition, $\mu_k(\sigma)$ is non-increasing and does not exceed $m(I_k)$. By (2)″ applied to I_k,

$$\begin{aligned} \mu_k(\sigma) &\leqq m(x|\ |u(x)-u_k| > \sigma - 2^n s \text{ in } I_k) \\ &\leqq e^{\alpha a} e^{-\alpha(\sigma-2^n s)} m(I_k). \end{aligned}$$

Then

$$\begin{aligned} s m(I_k) &\leqq \int_{I_k} |u| dx = \int_0^\infty \mu_k(\sigma) d\sigma \\ &= \int_0^{s/2} \mu_k(\sigma) d\sigma + \int_{s/2}^{2^n s} \mu_k(\sigma) d\sigma + \int_{2^n s}^\infty \mu_k(\sigma) d\sigma \\ &\leqq \frac{s}{2} m(I_k) + \left(2^n s - \frac{s}{2}\right) \mu_k\left(\frac{s}{2}\right) + \frac{1}{\alpha} e^{\alpha a} m(I_k). \end{aligned}$$

It follows for $s > \alpha^{-1} 2^{n+2} e^{\alpha a}$ that

$$\mu_k\left(\frac{s}{2}\right) \geqq \frac{1}{2^{n+1}}\, m(I_k) \geqq \frac{1}{2^{n+1}}\, e^{-\alpha a} e^{2^n \alpha s} \mu_k(2^{n+1} s).$$

Then also

$$\begin{aligned} m(S_{s/2}) &\geqq \sum_k \mu_k\left(\frac{s}{2}\right) \geqq 2^{-n-1} e^{-\alpha a} e^{2^n \alpha s} \sum_k \mu_k(2^{n+1} s) \\ &= 2^{-n-1} e^{-\alpha a} e^{2^n \alpha s} m(S_{2^{n+1}s}) \quad \text{for} \quad s > \frac{1}{\alpha}\, 2^{n+2} e^{\alpha a}. \end{aligned}$$

Inequality (2)‴ is an immediate consequence.

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Received February, 1961.