On Functions of Bounded Mean Oscillation*

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§ 1. We prove an inequality (Lemmas 1.1') which has been applied by one of the authors and by J. Moser in their papers in this issue. The inequality expresses that a function, which in every subcube C of a cube C₀ can be approximated in the $L^1$ mean by a constant $a_C$ with an error independent of C, differs then also in the $L^p$ mean from $a_C$ in C by an error of the same order of magnitude. More precisely, the measure of the set of points in C, where the function differs from $a_C$ by more than an amount $\sigma$ decreases exponentially as $\sigma$ increases.

In Section 2 we apply Lemma 1' to derive a result of Weiss and Zygmund [3], and in Section 3 we present an extension of Lemma 1'.

**Lemma 1.** Let $u(x)$ be an integrable function defined in a finite cube $C_0$ in $n$-dimensional space; $x = (x_1, \cdots, x_n)$. Assume that there is a constant $K$ such that for every parallel subcube $C$, and some constant $a_C$, the inequality

$$ \frac{1}{m(C)} \int_C |u - a_C| \, dx \leq K $$

(1)

holds. Here $dx$ denotes element of volume and $m(C)$ is the Lebesgue measure of C. Then, if $\mu(\sigma)$ is the measure of the set of points where $|u - a_{C_0}| > \sigma$, we have

$$ \mu(\sigma) \leq Be^{-b\sigma/k} m(C_0) \quad \text{for} \quad \sigma > 0, $$

(2)

where $B, b$ are constants depending only on $n$.

Since for every continuously differentiable function $f(s)$, vanishing at the origin,

$$ \int_{C_0} f(|u - a_{C_0}|) \, dx = \int_0^\infty \mu(s) \, df(s), $$

inequality (2) implies that $u$ belongs to $L^p$ for every finite $p \geq 1$, and, in fact for $b' < K^{-1} b$ the function $e^{b'|u - a_{C_0}|}$ is integrable and

$$ \int_{C_0} e^{b'|u - a_{C_0}|} \, dx \leq \left(1 + \frac{Bb'}{K^{-1}b - b'}\right) m(C_0). $$

(3)

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A function satisfying (1) for every subcube \( C \) of \( C_0 \), for some constant \( a_C \), will be said to have "mean oscillation \( \leq K \) in \( C_0 \)". Taking for \( a_C \) the average of \( u \) in \( C \) we always have

\[
\left( \frac{1}{m(C)} \int_C |u-a_C| \, dx \right)^2 \leq \frac{1}{m(C)} \int_C |u(x)-a_C|^2 \, dx = \frac{1}{3}(m(C))^{-2} \int_C \int_C dy \, u(x) - u(y)^2.
\]

In particular \( u \) has mean oscillation \( \leq K \), if \( u \) is bounded and its oscillation \( |u(x) - u(y)| \) does not exceed the value \( \sqrt{2} K \) in \( C_0 \).

Boundedness of \( u \) is not necessary for boundedness of its mean oscillation. Let indeed \( u(x) \) be any integrable function in \( C_0 \) with the property that we can associate with every subcube \( C \) a value \( a_C \) such that the subset \( S_\sigma \) of \( C \), where

\[
|u-a_C| \geq \sigma,
\]

has measure

\[
\mu(\sigma) \leq B e^{-b\sigma} m(C) \quad \text{for} \quad \sigma > 0.
\]

Then

\[
\int_C |u-a_C| \, dx = \int_0^\infty \mu(\sigma) \, d\sigma \leq \frac{B}{b} m(C)
\]

so that the mean oscillation of \( u \) does not exceed \( B/b \). Take now for \( u \) the function \( \log|x-y| \), where \( y \) is fixed. Let \( C \) be any cube of side \( h \), and let \( \xi \) and \( \eta \) be points of \( C \) for which

\[
|\xi-y| = \max_{x \in C} |x-y|, \quad |\eta-y| = \min_{x \in C} |x-y|.
\]

Take \( a_C = \log|\xi-y| \). Then for \( x \) in \( C \)

\[
|u(x)-a_C| = \log \frac{|\xi-y|}{|x-y|}.
\]

\( S_\sigma \) is the subset of \( C \) lying in the sphere

\[
|x-y| \leq |\xi-y| e^{-\sigma}.
\]

If \( S_\sigma \) is not empty, it must contain \( \eta \), so that

\[
|\xi-y| e^{-\sigma} \geq |\eta-y| \geq |\xi-y| - |\xi-\eta| \geq |\xi-y| - \sqrt{n} h.
\]

Thus

\[
|\xi-y| \leq \frac{\sqrt{n} h}{1-e^{-\sigma}}.
\]
It follows that $S_\sigma$ is contained in the sphere
\[ |x-y| \leq \frac{\sqrt{n} \, h}{e^\sigma - 1} \]
and that its measure $\mu(\sigma)$ does not exceed
\[ \left( \frac{\sqrt{n} \omega_n}{e^\sigma - 1} \right)^n m(C), \]
where the volume of the unit sphere in $n$-space is denoted by $(\omega_n)^n$. Since also $\mu(\sigma) \leq m(C)$ for all $\sigma \geq 0$, we find that
\[ \mu(\sigma) \leq (1 + \sqrt{n} \omega_n)^n e^{-n\sigma} m(C) = Be^{-b\sigma} m(C) \quad \text{for} \quad \sigma > 0, \]
where $B$ and $b$ do not depend on $C$. This proves that $\log|x-y|$ is of bounded mean oscillation in every cube $C_0$. The same holds then for any function $u(x)$ of the form
\[ u(x) = \int \xi(y) \log|x-y| dy \quad \text{with} \quad \int |\xi(y)| dy < \infty. \]

Lemma 1 will be derived from

\textbf{Lemma 1'.} Let $u(x)$ be integrable in a cube $C_0$ and assume that there is a constant $\kappa$ such that for every parallel subcube $C$ we have

\[ (1)' \quad \frac{1}{m(C)} \int_C |u-u_C| \, dx \leq \kappa, \]

where $u_C$ is the mean value of $u$ in $C$. Then if $S_\sigma$ is the set of points where $|u-u_C| > \sigma$, its measure $m(S_\sigma)$ satisfies

\[ (2)' \quad m(S_\sigma) \leq \frac{A}{\kappa} \int_{C_0} |u-u_{C_0}| \, dx \cdot e^{-a\sigma \kappa^{-1}} \quad \text{for} \quad \frac{\sigma}{\kappa} \geq a. \]

Since $m(S_\sigma) \leq m(C_0)$ it follows that

\[ (2)'' \quad m(S_\sigma) \leq e^{\alpha a} e^{-a \sigma \kappa^{-1}} m(C_0) \quad \text{for} \quad \sigma > 0. \]

Here $A \leq 1$, $\alpha$, $a$ are positive numbers depending only on the dimension $n$.

By a standard type of argument we can derive, as consequences of $(2)'$, the following inequalities: for $0 < \beta < \alpha$,

\[ (3)' \quad \int_{C_0} e^{\beta \kappa^{-1} |u-u_{C_0}|} \, dx \leq \left( \frac{\alpha}{\alpha - \beta} + e^{\beta a} \right) m(C_0), \]
\[
\int_{C_0} (e^{\beta x - |u-u_{C_0}|} - 1) \, dx \leq \left( \frac{e^{\beta a} - 1}{\beta a} + \frac{A}{\kappa} \frac{a}{\alpha - \beta} e^{a(\theta - \alpha)} \right) \int_{C_0} |u-u_{C_0}| \, dx
\]

(3)''

\[
= A \int_{C_0} |u-u_{C_0}| \, dx
\]

(3)'''

\[
\leq 2A \int_{C_0} |u| \, dx.
\]

The inequality (3)''' is of interest since, in case \( u(x) \) is integrable in an infinite cube \( C_0 \) and satisfies (1)' in every finite subcube, we can conclude from (3)''' that

(3)''''

\[
\int_{C_0} (e^{\beta x - |u|} - 1) \, dx \leq 2A \int_{C_0} |u| \, dx.
\]

If one wishes to prove (2)'''' directly, without proving (2)', then the proof given below can be simplified slightly.

Lemma 1 follows easily from Lemma 1'; for (1) implies that \( |u_C - a_C| \leq K \) so that

\[
\frac{1}{m(C)} \int_C |u-u_C| \, dx \leq 2K.
\]

By Lemma 1', (2)'' holds, with \( \kappa = 2K \), and (2) then follows easily.

The proof of Lemma 1' is based on a decomposition of integrable functions which, in one dimension, is due to F. Riesz, and which has been used extensively by Calderon and Zygmund [1] and Hörmander [2]. For completeness we include the proof of this decomposition, in a form suitable for application to Lemma 1'.

DECOMPOSITION. Let \( u \) be an integrable function defined in a cube \( C_0 \) and let \( s \) be a positive number such that

(4)

\[
s \geq \frac{1}{m(C_0)} \int_{C_0} |u| \, dx.
\]

There exists a denumerable number of open disjoint cubes \( I_k \) in \( C_0 \) such that

i) \( |u| \leq s \) a.e. in \( C_0 - \bigcup I_k \),

ii) the average value \( u_k \) of \( u \) in \( I_k \) is bounded in absolute value by \( 2^n s \),

iii) \( \sum I_k \)

\[
\sum m(I_k) \leq s^{-1} \int_{C_0} |u| \, dx.
\]

Proof: Divide \( C_0 \) (by halving each edge) into \( 2^n \) equal cubes and let \( I_{11}, I_{12}, \ldots \) be those open cubes over which the average value of \( |u| \) is \( \geq s \). Then

\[
s m(I_1) \leq \int_{I_1} |u| \, dx \leq 2^n s m(I_{1k})
\]
by (4). Next subdivide each remaining cube, over which the average of $|u|$ is $< s$, into $2^n$ equal cubes, and denote by $I_{s1}, I_{s2}, \ldots$ those cubes thus obtained over which the average of $|u|$ is $\geq s$. Again subdivide the remaining cubes, etc. In this way we obtain a sequence of cubes $I_{sk}$, which we rename $I_k$, such that

$$sm(I_k) \leq \int_{I_k} |u| dx < 2^n sm(I_k).$$

Clearly property ii) is satisfied. Furthermore, summing the left inequality over $k$ we obtain iii). We observe finally that a point of $C_0$ which does not belong to any of the $I_k$ belongs to arbitrarily small cubes over which the average of $|u|$ is $< s$. Hence $|u| \leq s$ a.e. outside all the $I_k$, verifying i).

Proof of Lemma 1': We may assume without loss of generality that $u c_0 = 0$ and that $\kappa = 1$, by replacing $u$ by $(u - uc_0)/\kappa$.

Denote by $F(\sigma)$ the smallest number, depending only on $\sigma$ and $n$ (and independent of the particular function $u$ or cube $C_0$) such that

$$m(S_\sigma) \leq F(\sigma) \int_{C_0} |u| dx;$$

obviously $F(\sigma) \leq 1/\sigma$. We now prove that, for $\sigma \geq 2^n$,

$$F(\sigma) \leq \frac{1}{s} F(\sigma - 2^n s) \quad \text{for} \quad 2^{-n} \sigma \geq s \geq 1. \quad (5)$$

To this end we apply the above decomposition to the function $u$ in $C_0$ with

$$2^{-n} \sigma \geq s \geq 1 \geq \frac{1}{m(C_0)} \int_{C_0} |u| dx,$$

the last inequality following from (1)''. Because of i) we see that if $|u(x)| > \sigma$, then $x$ belongs to one of the $I_k$ (except for a set of measure zero). Hence, since the average $u_k$ of $u$ in $I_k$ is bounded by $2^n s$ in absolute value, we see that

$$m(S_\sigma) = m\{x||u(x)| > \sigma\} \leq \sum_k m\{x||u(x) - u_k| > \sigma - 2^n s \ \text{in} \ I_k\}.$$

Now in the cube $I_k$ the function $u - u_k$ satisfies the hypotheses of Lemma 1', in particular it satisfies (1)'' for every cube in $I_k$. Hence, using the definition of $F(\sigma)$, we have

$$m\{x||u(x) - u_k| > \sigma - 2^n s \ \text{in} \ I_k\} \leq F(\sigma - 2^n s) \int_{I_k} |u - u_k| dx \leq F(\sigma - 2^n s) m(I_k).$$

Thus we find
\[ m(S_\sigma) \leq F(\sigma - 2^n s) \sum_k m(I_k) \leq \frac{1}{s} F(\sigma - 2^n s) \int_{C_\sigma} |u| dx \]

by iii), proving (5).

Setting \( s = e \) in (5) we see that if \( F(\sigma) \leq A e^{-\alpha \sigma}, \alpha = 1/(2^n e) \) for some \( \sigma \), then

\[ F(\sigma + 2^n e) \leq \frac{1}{e} A e^{-\alpha \sigma} = A e^{-\alpha (\sigma + 2^n e)}. \]

From this it follows that if on some interval of length \( 2^n e \) the inequality \( F(\sigma) \leq A e^{-\alpha \sigma} \) holds, then it holds for all larger \( \sigma \). But a calculation shows that

\[ F(\sigma) \leq \frac{1}{\sigma} \leq \frac{12}{10} 2^{-n} e^{-\alpha \sigma} \quad \text{for} \quad \frac{2^n e}{e-1} \leq \sigma \leq \frac{2^n e}{e-1} + 2^n e. \]

(This interval is the one of length \( 2^n e \) on which the maximum of \( e^{\alpha \sigma}/\sigma \) is as small as possible.) Thus we conclude that

\[ m(S_\sigma) \leq \frac{12}{10} 2^{-n} e^{-\alpha \sigma} \int_{C_\sigma} |u| dx \quad \text{for} \quad u \sigma \geq \frac{1}{e-1}, \quad \alpha = \frac{1}{2^n e}, \]

that is, we have proved (2)' with

\[ A = \frac{12}{10} 2^{-n}, \quad \alpha = \frac{1}{2^n e}, \quad \alpha = \frac{2^n e}{e-1}. \]

We have made no attempt here to obtain the best constants. The exponent \( \alpha \) can be considerably improved, i.e. increased, by using the hypothesis (1)' again to sharpen the estimate \( |u_k| \leq 2^n s \) that was provided by ii). We mention only that we have proved (2)' with a constant \( \alpha \) which for large \( n \) behaves like \( (1/e \log 2) (\log n/n) \).

§ 2. A recent paper by M. Weiss and A. Zygmund [3] contains the following

**Theorem.** If \( F(x) \) is periodic and for some \( \beta > \frac{1}{2} \) satisfies

\[ F(x+h) + F(x-h) - 2F(x) = O \left( \frac{1}{|\log h|^\beta} \right) \]

uniformly in \( x \), then \( F \) is the indefinite integral of an \( f \) belonging to every \( L_\beta \).

They also give an example showing that the result does not hold for \( \beta = \frac{1}{2} \).

The proof of the theorem in [3] is rather short but it relies on a theorem of Littlewood and Paley, and it seems of interest to us to show how it may be derived from our Lemma 1': we prove
LEMMA 2. Let \( u(x) \) be an integrable function defined in a finite cube \( C_0 \) in \( n \)-space. Assume that there is a constant \( K \) and a constant \( \beta > \frac{1}{2} \) such that if \( C_1 \) and \( C_2 \), are any two equal subcubes having a full \((n-1)\)-dimensional face in common, then

\[
|u_{C_1} - u_{C_2}| \leq \frac{K}{1 + |\log h|^\beta};
\]

here \( u_{C_1} \), \( u_{C_2} \) are the mean values of \( u \) in the cubes \( C_1 \) and \( C_2 \), and \( h \) is the common side length. Then \( u \) satisfies the conditions of Lemma 1' with some constant \( \kappa \) depending on \( K \), \( \beta \) and \( n \) so that, consequently, \( u \) satisfies (2)' or (3)'.

The preceding theorem follows easily from this lemma. By convolution of \( F \) with a smooth peaked kernel we may suppose that \( F \) is infinitely differentiable. It suffices merely to estimate the \( L_p \) norm of the derivative \( f \) of \( F \). Hypothesis (7) asserts simply that \( f \) satisfies (8) for \( n = 1 \). Applying Lemma 2 we obtain from (2)'' or (3)'' an estimate for the \( L_p \) norm of \( f \) depending only on \( K \) and \( \beta \), proving the theorem. From (3)'' we find, furthermore, that \( e^{a|x|} \) is integrable for some \( \alpha' > 0 \).

Proof of Lemma 2: Consider a subcube \( C_r \) of side length \( h \) subdivided into \( 2^{rn} \) equal cubes \( C_r, r = 1, \ldots, 2^{rn} \), obtained by dividing each edge into \( 2^N \) equal parts, and let \( u_r \) denote the mean value of \( u \) in \( C_r \). Then

\[
\frac{1}{m(C)} \int_C |u - u_C|dx = \lim_{N \to \infty} 2^{-rn} \sum_r |u_r - u_C|.
\]

Thus to prove (1)' it suffices to show that \( 2^{-rn} \sum_r |u_r - u_C| \leq \kappa, \) with \( \kappa \) depending only on \( K \), \( \beta \) and \( n \).

By Schwarz inequality,

\[
2^{-rn} \sum_r |u_r - u_C| \leq \left[2^{-rn} \sum_r |u_r - u_C|^2 \right]^{1/2} = a_N^{1/2}.
\]

We shall prove that the \( a_N \) are uniformly bounded by showing that

\[
a_{N+1} \leq a_N + \left( \frac{nK}{1 + |\log h|^\beta} \right)^2, \quad h = \frac{k}{2^{N+1}}.
\]

Since \( a_0 = 0 \), it follows that

\[
a_{N+1} \leq n^2K^2 \sum_{j=1}^\infty \left(1 + |j| \log 2 - \log h/\beta \right)^2 \leq \kappa^2
\]

for some constant \( \kappa \) independent of \( h \), convergence being guaranteed by the fact that \( \beta > \frac{1}{2} \).

Thus to complete the proof we shall establish (9). We observe first that

\[
2^{-rn} \sum_r u_r = u_C \quad \text{so that using the general identity}
\]
for real \( b_r \), we find
\[
2^{2nN} a_N = 2^n \sum_i |u_r - u_C|^2 = \frac{1}{2} \sum_{r,s} |u_r - u_s|^2.
\]

(10)

Now, on the next subdivision of \( C \) into \( 2^{n(N+1)} \) cubes each \( C_r \) is divided into \( 2^n \) equal cubes \( C_{ri}, i = 1, \ldots, 2^n \), of side length \( h = h/2^{N+1} \). If \( u_{ri} \) is the mean value of \( u \) in \( C_{ri} \) we have
\[
u_r = 2^{-n} \sum_i u_{ri}.
\]

(11)

Furthermore, since any two \( C_{ri}, C_{rj} \) can be connected by a chain of at most \( n+1 \) cubes each having a full face in common with the succeeding one, we find from (8) that
\[
|u_{ri} - u_r| \leq \frac{nK}{1+|\log h|^\beta} = M,
\]

where \( M \) is so defined. This together with (11) implies
\[
|u_{ri} - u_r| \leq M.
\]

According to formula (10)
\[
2^{2^n(n+1)} a_{N+1} = \sum_{r,s \leq 2^{nN}} |u_{r} - u_{s}|^2
\]
\[
= \sum_{i,j \leq 2^n} [(u_{ri}^2 + u_{sj}^2) - 2u_{ri}u_{sj}]
\]
\[
= \sum_{i,j} [(u_{ri} - u_r)^2 + (u_{sj} - u_s)^2 + 2u_{ri}u_s
\]
\[
+ 2u_{ri}u_s - u_r^2 - u_s^2 - 2u_{ri}u_{sj}]
\]
\[
= \sum_{i,j} [(u_{ri} - u_r)^2 + (u_{sj} - u_s)^2]
\]
\[
+ \sum_{r,s} [2^{2^nn} u_r^2 + u_s^2) - 2^{2n} (u_r^2 + u_s^2) - 2^{2n} u_r u_s],
\]

by (11),
\[
\leq 2M^2 2^{2^n(N+2n)} + 2^{2n} \sum_{r,s} (u_r - u_s)^2
\]
\[
= 2^{2^n(N+1)} M^2 + 2^{2^n(N+1)} a_N,
\]

by (10), or
\[
a_{N+1} \leq a_N + M^2.
\]

This is the desired inequality (9) and the proof is complete.
§ 3. In this section we present briefly a generalization of Lemma 1'.

**Lemma 3.** Let \( u \) be integrable in a finite cube \( C_0 \) and consider a subdivision of \( C_0 \) into a denumerable number of cubes \( C_i \), no two having a common interior point. Assume that for fixed \( \rho, \ 1 < \rho < \infty \), the expression

\[
\left( \sum_i m(C_i)^{1-p} \left( \int_{C_i} |u-u_{C_i}|^p dx \right)^{1/p} \right)
\]

is finite. Denote by \( K_u \) the lim sup of such expressions for all possible subdivisions of \( C_0 \) of this kind; in general \( K_u = \infty \). If \( K_u < \infty \), the measure \( m(S_\sigma) \) of the set \( S_\sigma \), where \( |u-u_{C_0}| > \sigma \), satisfies

\[
m(S_\sigma) \leq A \left( \frac{K_u}{\sigma} \right)^p \quad \text{for} \quad \sigma > 0,
\]

for some constant \( A \) depending only on \( n \) and \( \rho \).

The result implies that the function \( u \) belongs to \( L^p \) for every \( \rho' < \rho \).

For \( \rho = \infty \) the hypothesis of Lemma 3 agrees with that of Lemma 1'.

**Proof:** We shall not attempt to obtain the best constants. Let \( q = \rho/(\rho-1) \) be the conjugate exponent to \( \rho \). We may assume that \( u_{C_0} = 0 \). Using induction with respect to the integer \( j \) we shall prove that if

\[
(12) \quad s = \frac{2^{-n} \sigma}{\rho(q^j-1)+1} \leq \frac{K_u}{m(C_0)^{1/p}},
\]

then

\[
(13) \quad m(S_\sigma) \leq 2^{-nq^{1/q}+2/q^2+\cdots+jq^j} \left( \frac{2^n \rho(1-q^{-j})K_u}{\sigma} \right)^{p(1-1/q^j+1)} \left( \frac{1}{K_u} \int_{C_0} |u|^q dx \right)^{1/q^j}.
\]

Since

\[
m(S_\sigma) \leq \frac{1}{\sigma} \int_{C_0} |u| dx,
\]

(13) holds for \( j = 0 \). Suppose then it is true for \( j - 1 \), we wish to prove it for \( j \).

Since

\[
(14) \quad \frac{1}{m(C_0)} \int_{C_0} |u| dx \leq \frac{K_u}{m(C_0)^{1/p}},
\]

we may apply the decomposition of § 1 to \( u \), with \( s \) equal to its value in (12). Let \( u_k \) denote the mean value of \( u \) in \( I_k \), and set \( v_k = u - u_k \) in \( I_k \). From the definition of \( K_u \) we may assert that

\[
(15) \quad \sum_k K_u^2 \leq K_u^p.
\]
Setting $a_k = \int_{I_k} |v_k| dx$ we note further (as in (14)) that

$$m(I_k)^{1-p} a_k^p \leq K_{v_k}^p,$$

so that by Hölder’s inequality

$$\sum a_k \leq \left( \sum m(I_k)^{1-p} a_k^p \right)^{1/p} \sum m(I_k)^{1/q},$$

or

$$\sum a_k \leq K_u \left( S^{-1} \int_{C_0} |u| dx \right)^{1/q}$$

by (15) and iii).

As in the derivation of (5), we have

$$m(S_\sigma^c) \leq \sum_k m\{x \in I_k | v_k > \sigma - 2^n s \}.$$ 

Applying the induction hypothesis (13), for $i - 1$, to the functions $v_k$ in $I_k$ we find

$$m(S_\sigma^c) \leq \left[ 2^{n-q^{1/2} + \cdot \cdot \cdot + (i-1)/q^{i-1}} \left| \frac{2^n \rho (1 - q^{-i})}{\sigma - 2^n s} \right| q^{1-i/q^{i-1}} \right]$$

$$\cdot \sum K_{v_k}^{q^{1-i/q^{i-1}}} \left( \frac{1}{K_{v_k}} \int_{I_k} |v_k| dx \right)^{1/q^{i-1}}$$

$$\leq \left[ \sum K_{v_k}^{q^{1-i/q^{i-1}}} \right] \left( \sum K_{v_k}^{q^{1-i/q^{i-1}}} \right)^{1-1/q^{i-1}}$$

by Hölder’s inequality,

$$\leq \left[ \sum K_u^{2^{i-1}} \right] S^{-1} \int_{C_0} |u| dx \right)^{1/q^{i}} K_u^{p-\nu_{i+1}}$$

by (17) and (15), so that

$$m(S_\sigma^c) \leq \left[ \sum K_u^{2^{i-1}} \right] \left( S^{-1} \int_{C_0} |u| dx \right)^{1/q^{i}} K_u^{p-\nu_{i+1}}.$$

A slightly tedious calculation shows that this inequality is identical with the desired result (13).

Having established (13) we may now express it in a more convenient form: if (12) holds, then, in virtue of (14), there is a constant $k$ depending only on $n$ and $p$ such that

$$m(S_\sigma^c) \leq k \left( \frac{K_u}{\sigma} \right)^{q^{1-1/q^{i+1}}} m(C_0)^{1/q^{i+1}}.$$
or

\[ m(S_\sigma) \leq k \left( \frac{K_u}{\sigma} \right)^p \left| \frac{\sigma m(C_0)^{1/p}}{K_u} \right|^{q^{j+1}}. \]

If now \( 2^{-n} \sigma \geq K_u m(C_0)^{-1/p} \) and we choose the largest integer \( j \geq 0 \) so that (12) is satisfied, we have the opposite inequality for \( j+1 \):

\[ \frac{\sigma m(C_0)^{1/p}}{K_u} \leq 2^n (p(q^{j+1} - 1) + 1) \leq 2^n pq^{j+1}. \]

Inserting into the previous inequality we find

\[ m(S_\sigma) \leq k \left( \frac{K_u}{\sigma} \right)^p \left| 2^n pq^{j+1} \right|^{q^{j+1}} \]

\[ \leq A \left( \frac{K_u}{\sigma} \right)^p \text{ for } \sigma \geq 2^n K_u m(C_0)^{-1/p}, \]

for some constant \( A \) depending only on \( n \) and \( p \). Since \( m(S_\sigma) \leq m(C_0) \), the same inequality holds for all \( \sigma > 0 \), with some other constant \( A \), and the proof of the lemma is complete.

§ 4. Inequality (2)" in Lemma 1' can be replaced by the more general inequality

(2)" \[ m(S_{2^ns} \sigma) \leq Ae^{-B\sigma^{-1}} m(S_\sigma) \text{ for } \sigma > 0 \]

with \( A, B \) depending only on \( n \).

Proof: Let \( \kappa = 1 \), \( u_{\sigma} = 0 \). For a fixed positive \( s \) the cubes \( I_k \) shall be defined as in the proof of Lemma 1'. Put

\[ \mu_k(\sigma) = m(x \mid |u(x)| > \sigma \text{ in } I_k). \]

By definition, \( \mu_k(\sigma) \) is non-increasing and does not exceed \( m(I_k) \). By (2)" applied to \( I_k \),

\[ \mu_k(\sigma) \leq m(x \mid |u(x)| - u_{\sigma_k} > \sigma - 2^n s \text{ in } I_k) \]

\[ \leq e^{as} e^{-a(\sigma - 2^n s)} m(I_k). \]

Then

\[ sm(I_k) \leq \int_{I_k} |u| dx = \int_0^{\infty} \mu_k(\sigma) d\sigma \]

\[ = \int_0^{s/2} \mu_k(\sigma) d\sigma + \int_{s/2}^{2^n s} \mu_k(\sigma) d\sigma + \int_{2^n s}^{\infty} \mu_k(\sigma) d\sigma \]

\[ \leq \frac{s}{2} m(I_k) + \left( 2^n s - \frac{s}{2} \right) \mu_k \left( \frac{s}{2} \right) + \frac{1}{a} e^{as} m(I_k). \]

It follows for \( s > a^{-1} 2^{n+2} s^2 \) that
\[ \mu_k \left( \frac{s}{2} \right) \geq \frac{1}{2^{n+1}} m(I_k) \geq \frac{1}{2^{n+1}} e^{-\alpha a} e^{2^n \alpha s} \mu_k(2^{n+1}s). \]

Then also

\[ m(S_{s/2}) \geq \sum_k \mu_k \left( \frac{s}{2} \right) \geq 2^{-n-1} e^{-\alpha a} e^{2^n \alpha s} \sum_k \mu_k(2^{n+1}s) \]
\[ = 2^{-n-1} e^{-\alpha a} e^{2^n \alpha s} m(S_{2^{n+1}s}) \quad \text{for} \quad s > \frac{1}{\alpha} 2^{n+1} e^{\alpha a}. \]

Inequality (2)'' is an immediate consequence.

**Bibliography**


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