On $W^{1,p}$ Estimates for Elliptic Equations in Divergence Form

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To Eugene Fabes in memoriam

0 Introduction

We will study in this paper a method of approximation to obtain $W^{1,p}$ estimates for solutions to a large class of elliptic problems.

The general setting for the method will be the following:

(A) a regularity result for a fixed operator $\mathcal{A}_0$,

(B) a local estimate of solutions to the given $\mathcal{A}u = 0$ by comparison with solutions to $\mathcal{A}_0 u = 0$, and

(C) a real variable argument coming from the Calderón-Zygmund decomposition.

First, we will apply the method to study $W^{1,p}$ regularity for a nonlinear elliptic operator in divergence form. We would like to point out that in the particular case of a linear elliptic equation, this method gives an alternative proof to the classical one, which uses the general theory of singular integrals.

The utility of the method that we describe below is that hypotheses (A) and (B) are obtained directly by studying the *deviation of the "coefficients" of* $\mathcal{A}$ *from the "coefficients" of* $\mathcal{A}_0$, and this is usually not a difficult task.

A more interesting application of this kind of *approximation method* is to elliptic *homogenization* problems for which we obtain results that give $W^{1,p}$ estimates with a weak hypothesis of regularity in the coefficients.

For instance, we are able to study the following cases:

- the case of periodic, continuous coefficients, for which we obtain $W^{1,p}$ estimates for $p < \infty$,

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- elliptic transmission problems in two cases:
  - holes with a $\mathcal{C}^1$ boundary and then $W^{1,p}$ estimates for all $p < \infty$, and
  - holes with a Lipschitz boundary (in this case the homogenization problem satisfies $W^{1,p}$ estimates for some $2 < p < p_0$).

For the homogenization problems hypotheses (A) and (B) mean a *coarse bound* and a *limiting bound*, respectively. The first one gives information about the regularity of the solution, and the second one gives the information about the existence of a *corrector* in L^p. See [4] for details about homogenization problems and Section 4 for systematic definitions. L^p estimates in homogenization problems with coefficients $\mathcal{C}^\alpha$ can be seen in [2], where the results are obtained by estimating singular integrals.

The organization of the paper will be as follows: In the next section we will describe precisely the method by proving a general theorem of approximation. The study of some linear elliptic problems is the subject of Section 2. Section 3 will be devoted to the $W^{1,p}$ regularity of solutions of elliptic equations that can be approximated for convenient nonlinear operators. Finally, Section 4 contains the $W^{1,p}$ regularity result for homogenization problems.

We use the classical Hardy-Littlewood maximal operator, namely,

$$Mf(x) = \sup_{x\in Q,\, Q\,\text{cube}} \frac{1}{|Q|}\int_Q |f(y)|dy$$

which satisfies the $(1,1)$ weak-type inequality and obviously, by interpolation, the L^p-estimate (see, for instance, [8]).

1 $W^{1,p}$ Estimates by Approximation

In this section we study a general result of $W^{1,q}$ regularity under hypotheses that show the *philosophy* of the method in a transparent way. We begin with the statement of the general hypotheses.

(H1) REGULARITY FOR THE REFERENCE EQUATION. The solutions $u \in W^{1,p}$ to the equation

$$\operatorname{div} a_0(\nabla u) = 0 \tag{E1}$$

verifies for some constant B and all $Q \subset \Omega$

$$\|\nabla u\|^p_{L^\infty(Q)} \le \frac{B}{2}\frac{1}{|2Q|}\int_{2Q} |\nabla u|^p \, dx\,, \tag{1.1}$$

where $2Q$ is the double of Q. (That is, solutions to the problem $\operatorname{div} a_0(\nabla u) = 0$, naturally posed in the space $W^{1,p}$, enjoy interior $W^{1,\infty}$ regularity.)

H2. APPROXIMATION PROPERTY HYPOTHESIS. Let $u \in W^{1,p}$ be a solution to the equation

$$\operatorname{div} a(x, \nabla u) = 0\,. \tag{E2}$$

Then there exists a small $\varepsilon > 0$ such that for all $Q \subset \Omega$ the solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} a_0(\nabla u_h) = 0 & \text{in } Q \\ \qquad\qquad u_h = u & \text{on } \partial Q \end{cases} \tag{AP}$$

satisfies

(i) $\displaystyle \frac{1}{|Q|}\int_Q |\nabla u_h|^p\,dx \le \frac{1+\varepsilon}{|Q|}\int_Q |\nabla u|^p\,dx$

(ii) $\displaystyle \frac{1}{|Q|}\int_Q |\nabla (u-u_h)|^p\,dx \le \varepsilon^\alpha \frac{1}{|Q|}\int_Q |\nabla u|^p\,dx$ for some $\alpha > 0$.

The main result in this section is the following:

THEOREM A *Let q be a given real number, $q > p$. Let $u \in W^{1,p}$ be a solution to* (E2). *Assume that* (H1) *holds. Then there exists $\varepsilon_0 > 0$, $\varepsilon_0(q)$, such that if* (H2) *holds for some $0 < \varepsilon < \varepsilon_0$, then $u \in W^{1,q}$.*

We will use the following version of the Calderón-Zygmund decomposition result in our proof of Theorem A:

LEMMA 1.1 (Calderón-Zygmund) *Let Q be a bounded cube in $\mathbb{R}^N$ and $A \subset Q$ a measurable set satisfying*

$$0 < |A| < \delta |Q| \quad \textit{for some } 0 < \delta < 1\,.$$

Then there is a sequence of disjoint dyadic cubes obtained from Q, $\{Q_k\}_{k\in\mathbb{N}}$, such that

1. $|A - \cup Q_k| = 0$,
2. $|A \cap Q_k| > \delta |Q_k|$, *and*
3. $|A \cap \bar{Q}_k| < \delta |\bar{Q}_k|$ *if Q_k is a dyadic subdivision of $\bar{Q}_k$.*

PROOF: Divide Q into 2^N (Q_1^j) dyadic cubes. Choose those for which

$$|Q_1^j \cap A| > \delta |Q_1^j|\,.$$

Divide each cube that has not been chosen into 2^N dyadic cubes, $\{Q_2^j\}$, and repeat the process above iteratively. In this way we obtain a sequence of disjoint dyadic cubes, which we denote as $\{Q_k\}$. Now if $x \notin \bigcup_{k\in\mathbb{N}} Q_k$, then there exists a sequence of cubes $\{C_i(x)\}$ containing x with diameter $\delta(C_i(x)) \to 0$ as $i \to \infty$ and such that

$$|C_i(x) \cap A| < \delta |C_i(x)| < |C_i(x)|\,.$$

By the Lebesgue theorem we conclude that for almost every $x \in Q - \bigcup_{k\in\mathbb{N}} Q_k$, $x \in Q - A$. ∎

We will call a sequence like the one in Lemma 1.1 a *Calderón-Zygmund covering* for A. We would like point out that for each cube Q_k in a Calderón-Zygmund covering of A there exists a finite nested sequence of dyadic cubes

$$\tilde{Q}_k^1 \supset \tilde{Q}_k^2 \supset \cdots \supset \tilde{Q}_k^{r(k)} \supset Q_k\,, \quad l = 1, \ldots, r(k),$$

for which we have $|Q_k^l \cap A| \le \delta |Q_k^l|$. This finite family of dyadic cubes will be called the *chain of predecessors* of the cube Q_k. We will simply label the predecessor of Q_k, that is, the one in the previous dyadic step, $\tilde{Q}_k \equiv \tilde{Q}_k^{r(k)}$.

In fact, we will use the following consequence of Lemma 1.1.

LEMMA 1.2 *Let Q be a bounded cube in $\mathbb{R}^N$. Assume that A and B are measurable sets, $A \subset B \subset Q$, and that there exists a $\delta > 0$ such that*

(i) $|A| < \delta |Q|$ *and*

(ii) *for each Q_k dyadic cube obtained from Q such that $|A \cap Q_k| > \delta |Q_k|$, its predecessor $\tilde{Q}_k \subset B$.*

Then $|A| < \delta |B|$.

PROOF: Cover A with a Calderón-Zygmund covering. Extract a disjoint subcovering by the predecessor $\tilde{Q}_k$. From the hypothesis we have $|\tilde{Q}_k \cap A| \le \delta |\tilde{Q}_k|$ and $\tilde{Q}_k \subset B$. Hence

$$|A| = \sum_{k\in\mathbb{N}} |Q_k| \le \sum_{k\in\mathbb{N}} |A \cap \tilde{Q}_k| \le \delta |B|\,.$$

∎

The main point of the proof of Theorem A is the following result:

LEMMA 1.3 *Assume that* (H1) *holds. Let* $u \in W^{1,p}$ *be a solution of* (E2) *in* Q. *Denote by* A *the constant* $A = \max(2^N, 2^{p+1}B)$, *with* B *as in* (H1). *Then for* $0 < \delta < 1$ *fixed, there exists an* $\varepsilon = \varepsilon(\delta) > 0$ *such that if hypothesis* (H2) *holds for such* ε, *and* $Q_k \subset \bar{Q}_k \subset \frac{1}{4}Q$ *satisfies*

$$|Q_k \cap \{x \mid M(|\nabla u|^p) > A\lambda\}| > \delta |Q_k|\,, \tag{1.2}$$

the predecessor satisfies $\bar{Q}_k \subset \{x \mid M(|\nabla u|^p) > \lambda\}$. *(Remark: A does not depend on S.)*

PROOF: We argue by contradiction. If Q_k satisfies (1.2) and for the corresponding $\bar{Q}_k$ the conclusion is false, there exists an $x \in \bar{Q}_k$ such that

$$\frac{1}{|Q|}\int_Q |\nabla u(y)|^p \, dy \le \lambda \quad \text{for all cubes } Q \ni x\,.$$

Solving the corresponding problem (AP), namely,

$$\begin{cases} -\operatorname{div} a_0(\nabla u_h) = 0 & \text{in } \tilde{Q} = 4\bar{Q} \\ u_h = u & \text{on } \partial\tilde{Q}\,, \end{cases}$$

from (H1) and (H2) we have

$$\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |\nabla u_h(y)|^p \, dy \le \lambda$$

and as a consequence $\|\nabla u_h\|^p_{L^\infty(\bar{Q}_k)} \le \lambda \frac{B}{2}$. Moreover,

$$\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |\nabla (u - u_h)|^p \, dx \le \varepsilon^\alpha \lambda\,.$$

Consider the restricted maximal operator

$$M^*(|\nabla u(x)|^p) = \sup_{x \in Q,\, Q \subset 2\bar{Q}_k} \frac{1}{|Q|}\int_Q |\nabla u(y)|^p \, dy\,;$$

then for $x \in Q_k$, $M(|\nabla u(x)|^p) \le \max\{M^*(|\nabla u(x)|^p), 2^N\lambda\}$.

Since $A = \max\{2^N, 2^{p+1}B\}$, a direct computation proves that

$$
\begin{aligned}
&|\{x \in Q_k \mid M^*(|\nabla u|^p) \geq A\lambda\}| \\
&\quad \leq \left|\left\{x \in Q_k \mid M^*(|\nabla(u-u_h)|^p) + M^*(|\nabla(u_h)|^p) > \frac{A}{2^p}\lambda\right\}\right| \\
&\quad \leq \left|\left\{x \in Q_k \,|\, M^*(|\nabla(u-u_h)|^p) > \frac{A}{2^{p+1}}\lambda\right\}\right| \\
&\qquad + \left|\left\{x \in Q_k \mid M^*(|\nabla u_h|^p) > \frac{A}{2^{p+1}}\lambda\right\}\right| \\
&\quad = \left|\left\{x \in Q_k \mid M^*(|\nabla(u-u_h)|^p) > \frac{A}{2^{p+1}}\lambda\right\}\right| .
\end{aligned}
$$

Then by the $(1,1)$ weak-type inequality we obtain

$$
|\{x \in Q_k \mid M^*(|\nabla u|^p) \geq A\lambda\}| \leq C\frac{2^{p+1}}{A\lambda}\int_{\tilde{Q}} |\nabla(u-u_h)|^p \, dy \tag{1.3}
$$

or

$$
|\{x \in Q_k \mid M^*(|\nabla u|^p) \geq A\lambda\}| \leq c(N)\frac{2^{p+1}}{A}\varepsilon^\alpha |Q_k| \, . \tag{1.4}
$$

Then if $c(N)\dfrac{2^{p+1}}{A}\varepsilon^\alpha < \delta$ we reach a contradiction. ∎

Proof of Theorem A: Given $q > p$ we study when $g \equiv M(|\nabla u|^p) \in L^{q/p}$; by standard arguments of measure theory, $g \in L^{q/p}$ if and only if

$$
\sum_{k=1}^{\infty} A^{k\frac{q}{p}} \omega_g(A^k\lambda_0) < \infty \, , \tag{1.5}
$$

where ω_g is the distribution function of g (see [5]).

Now take A, δ, and the corresponding $\varepsilon > 0$ given by Lemma 1.3; by Lemma 1.2 we obtain that $\omega_g(A\lambda_0) \leq \delta\omega_g(\lambda_0)$ and by recurrence $\omega_g(A^k\lambda_0) \leq \delta^k\omega_g(\lambda_0)$. Then (1.5) implies that

$$
\sum_{k=1}^{\infty} A^{k\frac{q}{p}} \omega_g(A^k\lambda_0) \leq \omega_g(\lambda_0) \sum_{k=1}^{\infty} A^{k\frac{q}{p}} \delta^k \, .
$$

We need that $\delta A^{q/p} < 1$. If $M(|\nabla u|^p) \in L^{q/p}$ a fortiori $\nabla u \in L^q$. ∎

Remark. Assume the equation of reference satisfies the following weaker hypothesis of regularity:

(H1$'$). The solutions $u \in W^{1,p}$ to the equation

$$\operatorname{div} a_0(\nabla u) = 0$$

satisfies for some $q > p$ that there exists a constant B such that for any cube $Q \subset \Omega$

$$\left(\frac{1}{|Q|}\int_Q |\nabla u|^q\right)^{1/q} \leq \frac{B}{2}\left(\frac{1}{|2Q|}\int_{2Q} |\nabla u|^p dx\right)^{1/p} .$$

Assume that verifies a similar approximation property as (H2). Then we can obtain a $W^{1,s}$-estimate for $p < s < q$ in a similar way. In fact, (1.3) contains an extra term that can be handled by taking into account the weak type (r, r) estimate for the Hardy-Littlewood maximal operator for a convenient $r > 1$.

2 L^p Estimates for the Gradient of Linear Elliptic Equations in Divergence Form

We apply Theorem A to linear equations. This result can be obtained by the potential theory approach but the use of this method can be interesting in some applications. More precisely, consider the elliptic equation

$$D_i(a_{ij}(x)D_i u) = 0 \tag{E}$$

in some bounded domain Ω of $\mathbb{R}^N$ with

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for some $\lambda, \Lambda > 0$.

We then get the following result:

THEOREM B *Let p be a real number, $p > 2$. Then there exists $\varepsilon = \varepsilon(p) > 0$ such that if I is the identity matrix in $\mathbb{R}^N$ and*

$$\|I - a_{ij}\|_\infty \leq \varepsilon, \tag{HB}$$

then all solutions u to (E) *in $W^{1,z}$ satisfy $u \in W^{1,p}$.*

For the Laplacian we have the classical estimate

$$\|\nabla u\|_\infty^2 \leq C\frac{1}{|Q|}\int_Q |\nabla u(y)|^2\, dy\,.$$

Then to apply Theorem B we need the following lemma:

LEMMA 2.1 *Let $u \in W^{1,2}$ be a solution of* (E) *and assume that* (HB) *is satisfied. Then*

$$\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |\nabla(u-u_h)|^2\,dy \le \varepsilon^2 \frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |\nabla u|^2\,dy$$

where u_h is the solution to the problem

$$\text{(P)} \qquad \begin{cases} -\Delta u_h = 0 & \text{in } \tilde{Q} \\ \quad u_h = u & \text{on } \partial\tilde{Q}\,. \end{cases}$$

PROOF: Given $\varepsilon > 0$ by (HB) and integrating by parts, we have

$$\begin{aligned}
\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |\nabla(u-u_h)|^2\,dy &= \frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} \langle \nabla(u-u_h), \nabla(u-u_h)\rangle dy \\
&= \frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} \langle \nabla(u-u_h), a_{ij}\nabla u\rangle dy \\
&\quad - \frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} \langle \nabla(u-u_h), (I-a_{ij})\nabla u\rangle dy \\
&= \frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} \langle \nabla(u-u_h), (a_{ij}-I)\nabla u\rangle dy \\
&\le \varepsilon \left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |\nabla(u-u_h)|^2 dy\right)^{1/2} \left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |\nabla(u)|^2 dy\right)^{1/2}.
\end{aligned}$$

Hence we conclude that

$$\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |\nabla(u-u_h)|^2 dy \le \varepsilon^2 \frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |\nabla u|^2\,dy\,.$$

∎

COROLLARY 2.2 *If we assume that the equation* (E) *has continuous coefficients, then each solution $u \in W^{1,2}$ verifies that $u \in W^{1,p}$ for all $p < \infty$.*

3 L^p Estimates for the Gradient of Nonlinear Elliptic Equations in Divergence Form

We study in this section a more general model of nonlinear elliptic equations. More precisely, consider

$$a : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N\,,$$

where $a(x,\xi)$ is a Carathéodory function, namely, measurable in x and derivable with respect to ξ for fixed x. Assume, moreover, the following:

1. $a(x,0)=0$.

2. $\langle D_\eta a(x,\eta)\xi,\xi\rangle \geq \gamma(\kappa+|\eta|^{p-2})|\xi|^2$. Here D_η is the Jacobian matrix of a with respect to η.

3. $\|D_\eta a(x,\eta)\| \leq \Gamma(\kappa+|\eta|^{p-2})$.

Here γ, κ, and Γ are positive constants and κ can be zero (degenerate case). Under these hypotheses we can find the following inequalities:

$$\begin{aligned}&\langle a(x,\eta)-a(x,\eta'),(\eta-\eta')\rangle\\ &\geq \gamma\begin{cases}|\eta-\eta'|^p & \text{if } p\geq 2\\ |\eta-\eta'|^2(1+|\eta|+|\eta'|)^{p-2} & \text{if } 1<p\leq 2\,.\end{cases}\end{aligned} \tag{3.1}$$

See, for instance, [10].

Consider the equation

$$\operatorname{div} a(x,\nabla u)=0 \tag{EQ}$$

and $u\in W^{1,p}$ a solution to (EQ). The method developed in Section 1 allows us to show the following result:

THEOREM C *Let q be a real number, $q>p$; then there exists $\varepsilon>0$ such that if*

$$\||\xi|^{p-2}\xi - a(x,\xi)\| \leq \varepsilon|\xi|^{p-1}\,, \tag{HC}$$

then all solutions $u\in W^{1,p}$ to (EQ) *verifies $u\in W^{1,q}$.*

We need to check in detail the inequality for the gradient of a p-harmonic function, namely, hypothesis (H1) and the approximation by p-harmonic functions that is the actual meaning of hypothesis (H2).

LEMMA 3.1 *Consider u a p-harmonic function; if Q and $2Q$ are concentric cubes related for a factor 2, then*

$$\|\nabla u\|^p_{L^\infty(Q)} \leq C(p,N)\frac{1}{|2Q|}\int_{2Q}|\nabla u|^p\,dx\,.$$

PROOF: In the case $p=2$ we have directly that the gradient of a solution is a solution and then a superlinear power is a subsolution. If $p\neq 2$ a direct proof can be found in [3, proposition 3, p. 838]. ∎

LEMMA 3.2 *Assume $u \in W^{1,p}$ is a solution of* (EQ) *such that in some cube Q*

$$\frac{1}{|Q|}\int_Q |\nabla u|^p \, dy \le \lambda \,,$$

and assume u_h is the solution of the nonlinear problem

$$\text{(PQ)} \qquad \begin{cases} -\Delta_p u_h \equiv -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } Q \\ \qquad\qquad\qquad\qquad\qquad u_h = u & \text{on } \partial Q \,, \end{cases}$$

and that (HC) *in Theorem* C *is satisfied for some $\varepsilon > 0$. Then*

$$\frac{1}{|Q|}\int_Q |\nabla u - \nabla u_h|^p \, dy \le \gamma^{-1}\varepsilon^{\alpha}\lambda$$

where $\alpha = p/(p-1)$ if $p \ge 2$ and $\alpha = p$ if $1 < p \le 2$.

PROOF: (i) $p \ge 2$. Call $a_p = \gamma^{-1}$. Then taking into account inequality (3.1) and equation (PQ) and then integrating by parts, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |\nabla(u-u_h)|^p \, dy \\
&\quad \le a_p \frac{1}{|Q|}\int_Q \langle(-\Delta_p u + \Delta_p u_h), (u-u_h)\rangle dy \\
&\quad = a_p \frac{1}{|Q|}\int_Q \langle |\nabla u|^{p-2}\nabla u, \nabla(u-u_h)\rangle dy \\
&\quad = \frac{a_p}{|Q|}\left(\int_Q \langle(|\nabla u|^{p-2}\nabla u - a(x,\nabla u), \nabla(u-u_h)\rangle dy \right.\\
&\qquad\qquad \left. + \int_Q \langle (a(x,\nabla u), \nabla(u-u_h)\rangle dy\right) \\
&\quad = a_p \frac{1}{|Q|}\int_Q \langle(|\nabla u|^{p-2}\nabla u - a(x,\nabla u), \nabla(u-u_h)\rangle dy \\
&\quad \le a_p\varepsilon\left(\frac{1}{|Q|}\int_Q |\nabla(u)|^p dy\right)^{(p-1)/p}\left(\frac{1}{|Q|}\int_Q |\nabla u - \nabla u_h|^p dy\right)^{1/p}
\end{aligned}$$

by hypothesis (HC). Then we conclude that

$$\frac{1}{|Q|}\int_Q |\nabla u - \nabla u_h|^p dy \le \gamma^{-1}\varepsilon^{p/(p-1)}\lambda \,.$$

(ii) $1 < p \le 2$. From the second inequality in (3.1) we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q (1 + |\nabla u| + |\nabla u_h|)^{p-2} \, |\nabla(u-u_h)|^2 \, dy \\
&\qquad \le \gamma \frac{1}{|Q|}\int_Q \langle(-\Delta_p u + \Delta_p u_h), (u-u_h)\rangle \, dy \,.
\end{aligned}$$

Then by the Hölder inequality,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |\nabla(u-u_h)|^p\,dy \\
&\quad\le \left(\frac{1}{|Q|}\int_Q (1+|\nabla u|+|\nabla u_h|)^{p-2}\,|\nabla(u-u_h)|^2 dy\right)^{\frac{p}{2}} \\
&\qquad\times\left(\frac{1}{|Q|}\int_Q (1+|\nabla u|+|\nabla u_h|)^p\,dy\right)^{\frac{(2-p)}{2}} \\
&\quad\le C\left(\gamma\frac{1}{|Q|}\int_Q \langle(-\Delta_p u+\Delta_p u_h),(u-u_h)\rangle dy\right)^{p/2}\lambda^{(2-p)/2},
\end{aligned}$$

and by the same argument as in the case $p \ge 2$, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |\nabla(u-u_h)|^p dy \\
&\quad\le C\left(\varepsilon(\frac{1}{|Q|}\int_Q |\nabla(u-u_h)|^p dy)^{1/p}\frac{1}{|Q|}\int_Q |\nabla u|^p dy)^{(p-1)/p}\right)^{p/2} \\
&\qquad\times\ \lambda^{(2-p)/2}.
\end{aligned}$$

Then

$$\left(\frac{1}{|Q|}\int_Q |\nabla(u-u_h)|^p dy\right)^{1/2} \le C\varepsilon^{p/2}\lambda^{1/2}$$

and we are done. ∎

As a consequence we have the following result:

COROLLARY 3.3 *Assume that the vector field $a(x,\xi)$ is continuous in x; then each solution $u \in W^{1,p}$ to the equation* (EQ) *belongs to $W^{1,q}$ for all $q < \infty$.*

4 Regularity for Homogenization Problems

It is clear from the previous sections that we really do not need the function u to satisfy an equation; all we need is for u to be close in "energy" and at every scale to a function (or vector, in the case of systems) that locally lies in a better functional space. We illustrate this with the theory of homogenization. Consider the matrix

$$A(y) = (a_{ij}(y))_{i,j=1,\dots,N}$$

satisfying

- $(a_{ij}(y))_{i,j=1\dots N}$ is T-periodic, $T \in \mathbb{R}^N$,
- $a_{ij}(y) \in L^\infty(\mathbb{R}^N)$, and
- $\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$ for some $0 < \lambda < \Lambda$.

We will consider solutions of the elliptic problems

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} -\operatorname{div}\left(A\left(\dfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = 0 \text{ in } Q \\ u_\varepsilon \text{ satisfying boundary conditions on } \partial Q \end{cases}$$

where $\varepsilon > 0$ is a small parameter and Q is a bounded cube in $\mathbb{R}^N$.

It is well-known that solutions u_ε to (P$_\varepsilon$) converge weakly in the Sobolev space $W^{1,2}$ to u_0, which is a solution to the constant-coefficient elliptic problem

$$(\mathrm{P}_0) \qquad \begin{cases} -\operatorname{div}\left(\tilde{A}\nabla u_0\right) = 0 \text{ in } Q \\ u_0 \text{ satisfying boundary conditions on } \partial Q\,, \end{cases}$$

where the entries of $\tilde{A}$ are given by

$$\tilde{a}_{il} \equiv \int_T a_{ij}(y)(\delta_{jl} + w^l_{y_j})dy\,, \quad 1 \le i, l \le N,$$

and w^l is the solution to the adjoint corrector problem

$$(\mathrm{P}_c) \qquad \begin{cases} -\left(a_{ij}(y)w^l_{y_j}\right)_{y_i} = (a_{il}(y))_{y_i} \text{ in } \mathbb{R}^N \\ w^l \ T\text{-periodic}, \quad 1 \le l \le N\,. \end{cases}$$

The corrector measures the defect to strong convergence by giving the asymptotic behavior of the oscillations in a convenient norm. See [4] for details. Bounds in $\mathcal{C}^{1\alpha}$, uniformly in ε, are obtained in [1] under the hypothesis that A is Hölder continuous. We will study in this section uniform $W^{1,p}$ estimates in two cases:

- $A(y)$ continuous and
- transmission problems with
  - $\mathcal{C}^1$ holes in a cube Q or
  - Lipschitz holes in a cube Q.

4.1 $A(y)$ Continuous

According to Section 2, if $A(y)$ is continuous, then we should expect $W^{1,p}$ estimates for all p, $1 < p < \infty$. We will assume general hypotheses that contain the above homogenization problem.

Statement of the Hypotheses

We will consider a one-parameter family of functions $F_\varepsilon \subset W^{1,2}(Q)$ (the "solutions"), with the following renormalization properties ($0 < \varepsilon \leq 1$):

(h1) (COARSE BOUND) If $u_\varepsilon \in F_\varepsilon$, $u_\varepsilon(\delta x)|_Q \in F_{\varepsilon/\delta}$, and if $u_1 \in F_1$,

$$\|\nabla u_1\|_{L^\infty(Q_{1/2})} \leq C\|\nabla u\|_{L^2(Q_1)} .$$

(h2) (LIMITING BOUND) There exists a universal constant M such that

$$\Big|\{x \mid |\nabla u_\varepsilon| > M\} \cap Q_{1/2}\Big| \leq D(\varepsilon)\|\nabla u\|^2_{L^2(Q_1)}$$
$$\text{and } D(\varepsilon) \to 0 \text{ as } \varepsilon \to 0 .$$

The main theorem in this case is the following:

THEOREM D *Assume that the solutions to problems* (P_ε) *satisfies* (h1) *and* (h2)*. Then for all* $p \in (1, \infty)$ *there exists a* $C(p)$ *such that, independently of* ε*,*

$$\|\nabla u_\varepsilon\|_{L^p(Q_{1/2})} \leq C(p)$$

provided that

$$\int_{Q_1} |\nabla u_\varepsilon|^2 dx \leq 1 .$$

Sketch of the Ideas

First, we sketch the idea of the proof, and then we prove the results as lemmas.

Consider the maximal function $\mathcal{M}_\varepsilon(x) \equiv M(|\nabla u_\varepsilon(x)|^2)$ defined above. We want to show that given $\delta > 0$,

$$|\{x \mid \mathcal{M}_\varepsilon(x) > \lambda\}| \leq C(\delta)\lambda^{-(1/\delta)} ,$$

because now the problem is to obtain the estimates for all $p < \infty$. If we get the previous estimate for the distribution function of the Hardy-Littlewood maximal operator, then we can proceed in the same way as in the proof of

Theorem A. The insight is the following: Given a finite p we choose a corresponding μ for which the summability in the Stieltjes sums for the L^p-norm of the maximal function can be guaranteed. This can be done by hypothesis (h2), and we get, for instance,

$$|\{x \in Q_1 \mid \mathcal{M}_\varepsilon(x) > M\}| \leq \mu \quad \text{if } \varepsilon \leq \varepsilon_0\,. \tag{4.1}$$

Then the idea is to study the sets

$$A_k(\varepsilon) = \left\{x \in Q_1 \mid \mathcal{M}_\varepsilon(x) > 20^n M^k\right\}. \tag{4.2}$$

Take the Calderón-Zygmund covering for $A_k(\varepsilon)$ with $\delta = 20^n\mu$ and $\{Q_j^k\}$, and call s_j the side of Q_j^k. Let $\varepsilon_0 = 2^{-l_0}$. We classify the cubes in the following way:

1. For those Q_j^k verifying that $\varepsilon_j = \varepsilon/s_j$ is such that $\varepsilon_j \leq \varepsilon_0$, we put $A_k(\varepsilon) \cap Q_j^k$ as a part of the set B_k. Then

   $$B_k = \bigcup_{\varepsilon_j \leq \varepsilon_0} (A_k(\varepsilon) \cap Q_j^k)\,. \tag{4.3}$$

   Hence B_k contains the part of $A_k(\varepsilon)$ corresponding to the cubes of high frequencies.

2. For those Q_j^k verifying that $2^{-(l+1)} \leq \varepsilon_j \leq 2^{-l}$ for $1 \leq l < l_0$, we put $A_k(\varepsilon) \cap Q_j^k$ as a part of the set $C_k^l(\varepsilon)$. Namely,

   $$C_k^l(\varepsilon) = \bigcup_{2^{-(l+1)} \leq \varepsilon_j \leq 2^{-l}} (A_k(\varepsilon) \cap Q_j^k)\,. \tag{4.4}$$

If we call ε_0 the critical scale, while $B_k(\varepsilon)$ represents the cubes with subcritical scale or, equivalently, high frequencies, the sets $C_k^l(\varepsilon)$ contain the cubes with supercritical scales, or low frequencies, classified by levels.

The measure of $B_k(\varepsilon)$ will be estimated with the same type of arguments as in Section 1 as we will see in the next result (Lemma 4.1) and its corollary.

The estimate of the measure of the sets $C_k^l(\varepsilon)$ will be reduced to the estimate of the measure of $B_{k-m}(\varepsilon)$ where m depends only on ε_0, namely, to the sets of high frequencies of a previous step $k - m$.

LEMMA 4.1 *Let $B_k(\varepsilon)$ and $B_{k+1}(\varepsilon)$ be defined as in* (4.3). *Then*

$$|B_{k+1}(\varepsilon)| \leq 20^n \mu |B_k(\varepsilon)|\,. \tag{4.5}$$

PROOF: We try to apply Lemma 1.2 with $A \equiv B_{k+1}(\varepsilon)$ and $B \equiv B_k(\varepsilon)$. If Q is a cube of the Calderón-Zygmund covering of $B_{k+1}(\varepsilon)$ for $20^n\mu$, then, by definition, it is also a cube of the Calderón-Zygmund covering of $A_{k+1}(\varepsilon)$. Let $\tilde{Q}$ be the predecessor of Q and s be the side of $\tilde{Q}$. We will prove that $\tilde{Q} \subset A_k(\varepsilon)$ and equivalently that $\tilde{Q} \subset B_k(\varepsilon)$, since $\tilde{Q}$, being of higher frequency than Q, is also of high frequency. Assume the contrary, i.e., there exists $x_0 \in \tilde{Q}$ such that

$$\mathcal{M}_\varepsilon(x_0) \leq M^k.$$

Scaling by $\bar{u}_\varepsilon(y) = \dfrac{u_\varepsilon(sx)}{M^k}$, $\tilde{Q}$ becames Q_1 and we have

$$\int_{Q_1} |\nabla u_\varepsilon|^2 dy \leq \frac{\mathcal{M}_\varepsilon(x_0)}{M^k} \leq 1\,.$$

Then by the definition of μ in (4.1) we have that

$$\Big|\{x \in Q_{1/2} \mid \mathcal{M}\bar{u}_\varepsilon(x) \geq M\}\Big| < \mu\,,$$

but this contradicts the fact that $|A_{k+1}(\varepsilon) \cap Q| \geq 20^n\mu|Q|$. ∎

COROLLARY 4.2 *If B_k and μ are defined by* (4.3) *and* (4.1)*, respectively, then $|B_k| \leq \mu^k$.*

LEMMA 4.3 *Let $C_k^l(\varepsilon)$ be defined by* (4.4)*. There exists a constant $m = m(\varepsilon_0)$ such that for any $l < l_0$,*

$$C_k^l(\varepsilon) \subset B_{k-m}(\varepsilon)\,.$$

PROOF: Let $Q_i \cap A_k(\varepsilon)$ be a part of $C_k^l(\varepsilon)$. Consider the chain of predecessors of Q_i, $\tilde{Q}_i^1, \ldots, \tilde{Q}_i^{r(i)}$, and the measure of the intersections $\tilde{Q}_i^t \cap A_{k-m}$. Take the biggest $\tilde{Q}_i^t$ for which

$$\left|\tilde{Q}_i^t \cap A_{k-m}\right| > 20^n\mu\left|\tilde{Q}_i^t\right| \tag{4.6}$$

The corresponding scale for $\tilde{Q}_i^t$ is supercritical, namely,

$$\bar{\varepsilon}(\tilde{Q}_i^t) = \frac{\varepsilon}{s(\tilde{Q}_i^t)} \geq \varepsilon_0$$

where $s(\tilde{Q}_i^t)$ is the side length.

Then $\tilde{Q}_i^t$ must be contained in $A_{k-m}(\varepsilon)$. If that were not the case, $\tilde{Q}_i^{t-1}$ would not be contained in $A_{k-m+1}(\varepsilon)$ and then, for some $x_0 \in \tilde{Q}_i^{t-1}$, $\mathcal{M}_\varepsilon(x_0) \le M^{k-m+1}$; in particular,

$$\frac{1}{|\tilde{Q}_i^{t-1}|}\int_{\tilde{Q}_i^{t-1}} |\nabla u_\varepsilon|^2\, dy \le M^{k-m+1}$$

and since $\tilde{Q}_i^{t-1}$ has supercritical scale, by rescaling, the hypothesis (h1) gives that

$$\|\nabla u_\varepsilon\|_{L^\infty(\tilde{Q}_i^t)} \le C(\varepsilon_0) M^{k-m+1}\,.$$

Hence choosing $m = m(\varepsilon_0)$ in such way that $C(\varepsilon_0)M^{k-m+1} \le M^k$, we get a contradiction.

Therefore choosing $m = m(\varepsilon_0)$ as above we have that the first predecessor for which

$$|\tilde{Q}_i^t \cap A_{k-m(\varepsilon_0)}(\varepsilon)| > \mu|\tilde{Q}_i^t|$$

has scale $\bar{\varepsilon} \le \varepsilon_0$; namely, $\tilde{Q}_i^t \cap A_{k-m(\varepsilon_0)}(\varepsilon)$ is a part of $B_{k-m(\varepsilon_0)}(\varepsilon)$. ∎

PROOF OF THEOREM D: We have that for all $\varepsilon > 0$ fixed, the distribution function of the maximal operator $\mathcal{M}_\varepsilon$ satisfies

$$|A_k(\varepsilon)| \le |B_k| + |B_{k-m(\varepsilon_0)}| \le \mu^k + \mu^{k-m(\varepsilon_0)}\,.$$

Then given p we choose μ and finish the proof as in Section 1. ∎

4.2 Transmission Problems

Finally, we will study the homogenization of a transmission problem. More precisely, we assume

$$A(x) = \sum_{i=1}^{r} d_i \chi_{D_i}(x) \tag{4.7}$$

where Q is a cube and $\bar{D}_i \subset Q$ are bounded domains in $\mathbb{R}^N$ with boundaries

- $\mathcal{C}^1$ in the first case and
- Lipschitz in the second case.

We will consider $A(y)$ extended periodically to the whole $\mathbb{R}^N$, and we also assume the ellipticity hypothesis.

Consider the problems

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} -\operatorname{div}\left(A\left(\dfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = 0 \text{ in } \Omega \\ u_\varepsilon \text{ satisfying boundary conditions on } \partial\Omega\,. \end{cases}$$

In the case of $\mathcal{C}^1$ boundaries, if $u_\varepsilon \in W^{1,2}$, then $u_\varepsilon \in W^{1,p}$ for all $p < \infty$. This regularity result is a consequence of the potential theory results by Fabes, Jodeit, and Rivière. See [7] for details.

According to [6], in the case of Lipschitz boundaries, we have that the solutions are in $W^{3/2,2}$. See also [9] for other references about the regularity in Lipschitz domains.

Hence in both cases we have $W^{1,p}$ regularity for $2 < p < p_0$; in case 1, $p_0 = \infty$, while in case 2, $p_0 = 2N/(N-1)$. In this way and also by using the definition of the correctors in [4], we get the situation described below.

4.3 Statement of the Hypotheses

As before, we have a one-parameter family of functions F_ε in $W^{1,2}(Q)$ with the same renormalization properties, but now we will assume that

(i) $\|\nabla u_1\|_{L^p(Q_{1/2})} \leq C\|\nabla u_1\|_{L^2(Q_1)}$ for some $p > 2$, and

(ii) there exists a universal constant C_0 such that

$$\left|\{M(|\nabla u_\varepsilon|^2 > \lambda^2\}\right| \leq C_0(\lambda^{-p} + \sigma(\varepsilon)\lambda^{-2})$$
$$\text{where } \sigma(\varepsilon) \to 0 \text{ as } \varepsilon \to 0\,.$$

Remark. For the homogenization problem, the *coarse bound* in (i) is given by the regularity for the problem, while the *limiting bound* in (ii) is given by the existence of correctors in $W^{1,p}$.

Using the same line of ideas used in Theorem D, we have the following result:

THEOREM E *Assume that the solutions to problems* (P_ε) *satisfy* (i) *and* (ii). *Then for all $q < p$ there exists a constant C_q independent of ε such that*

$$\|\nabla u_\varepsilon\|_{L^q(Q_{1/2})} < C_q\|\nabla u_\varepsilon\|_{L^2(Q_{1/2})}\,.$$

PROOF: We prove that for $2 < q < p$ fixed and by choosing c_1 and M large enough

$$|A_k(\varepsilon)| = \left|\{M(|\nabla u_\varepsilon|^2) \geq c_1 M^{2k}\}\right| \leq CM^{-qk}\,, \tag{4.8}$$

and then we finish as at the end of the proof of Theorem A. We first choose M so that

$$C_0 M^{-p} \leq \frac{1}{10^n} M^{-q}$$

and then

$$\varepsilon_0 = 2^{-k_0} \quad \text{for which} \quad C_0 \sigma(\varepsilon) M^{-2} \leq \frac{1}{10^n} M^{-q} \quad \text{for } \varepsilon < \varepsilon_0\,.$$

For $\delta = \frac{1}{2^n} M^{-p/2}$ we consider the Calderón-Zygmund covering of $A_k(\varepsilon)$. As in the proof of Theorem D we split $A_k(\varepsilon)$ in the part of high frequencies, B_k, and the part of low frequencies classified by its level, namely,

$$A_k(\varepsilon) = B_k \cup \left(\bigcup_{1 \leq l \leq k_0} C_k^l \right)$$

where if Q_j^k is a cube in the Calderón-Zygmund decomposition and s_j its side, then

- B_k is the subset of $A_k(\varepsilon)$ contained in the cubes Q_j^k for which $\varepsilon/s_j \leq \varepsilon_0 = 2^{-k_0}$, and
- C_k^l is the subset of $A_k(\varepsilon)$ contained in the cubes Q_j^k for which $\varepsilon_0 2^{l-1} \leq \varepsilon/s_j \leq \varepsilon_0 = 2^l$.

Now a predecessor, $\tilde{Q}$, of a cube, Q, defining B_k is a fortiori in the range of high frequency, because it has a larger side, $\bar{s}_j$, than Q and then $\varepsilon/\bar{s}_j \leq \varepsilon_0$. Therefore the choice of ε_0 and M and the argument in Lemma 4.1 and its corollary imply that

$$|B_k| \leq \frac{1}{2^n} M^{-kq}\,.$$

We now choose $m = m(\varepsilon_0)$ such that

$$\frac{C}{\varepsilon_0} M^{k-m} \leq M^k\,.$$

Then we have the following claim:

Claim. If Q_k^j belongs to the Calderón-Zygmund covering of A_k, then its predecessor is contained in A_{k-m} and, in particular, can be covered with cubes in the Calderón-Zygmund decomposition of A_{k-m}.

From the claim taking into account the size of the cubes, it follows that

$$C_k^l \subset B_{k-m} \cup \left(\bigcup_{s<l} C_{k-m}^s \right);$$

thus

$$|C_l^1| \leq \frac{1}{2^n} M^{-(k-m)q}, \quad |C_l^2| \leq \frac{1}{2^n} M^{-(k-2m)q}, \quad \cdots \quad .$$

Then by choosing n large enough,

$$|A_k| \leq \bar{C} M^{-kq} \quad \text{where} \quad \bar{C} = M^{-k_0 mq}.$$

We point out that in the first step, $k_0 m$, we get the inequality by using the uniform weak-type estimates in the unit cube, namely,

$$\begin{aligned} |A_{k_0 m}| = \left| \{ M(|\nabla u_\varepsilon|^2) \geq C_1 M^{2k_0 m} \} \right| &\leq c \frac{1}{C_1 M^{2k_0 m}} \int_{Q_1} |\nabla u_\varepsilon|^2 \, dx \\ &\leq c \frac{1}{C_1 M^{2k_0 m}} . \end{aligned}$$

Hence by choosing C_1 large we get the inequality also for the first case and then we finish as in Theorem D.

It remains to justify the claim. We will use an argument by contradiction. If we assume that Q_k^j is in the Calderón-Zygmund decomposition of A_k but its predecessor $\bar{Q}_k^j \not\subseteq A_{k-m}$, we can find $x_0 \in \bar{Q}_k^j$ such that $M(|\nabla u_\varepsilon(x_0)|^2) < C_1 M^{2(k-m)}$ in particular, and by hypothesis (i) we get

$$\frac{1}{|\bar{Q}_k^j|} \int_{\bar{Q}_k^j} |\nabla u_\varepsilon|^p dx \leq M^{pk}$$

and by scaling

$$\int_{Q_1} |\nabla \bar{u}_\varepsilon|^p dx \leq 1 .$$

Scaling again, we obtain

$$\left| Q_k^j \cap \{ M(|\nabla u_\varepsilon|^2) > C_1 M^{2k} \} \right| \leq C M^{-p} |Q_k^j| ,$$

which contradicts the choice of δ and the hypothesis. ∎

Remark. Consider the problem

$$\text{(TP)} \qquad \begin{cases} \mathcal{L}(u) \equiv -\operatorname{div}(A(x)\nabla u) = \operatorname{div} f & \text{in } \Omega,\ f \in L^r,\ r > N \\ \text{boundary condition on } \partial\Omega\,, \end{cases}$$

where we assume that $A(x)$ is a continuous matrix unless in a $\mathcal{C}^1$ or a Lipschitz surface Σ that separates two subdomains Ω_1 and Ω_2 of Ω, namely, $\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma$ and

$$A(x) = \begin{cases} (a^1_{i,j}(x)) & \text{if } x \in \Omega_1 \\ (a^2_{i,j}(x)) & \text{if } x \in \Omega_2\,. \end{cases}$$

$(a^1_{i,j}(x))$ and $(a^2_{i,j}(x))$ are continuous, and moreover

$$\nu|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \frac{1}{\nu}|\xi|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^N\,.$$

The regularity of the solution depends on f and in the regularity of the coefficients of the matrix A. We will isolate the problem when $f = 0$. Taking into account the regularity results in [6, 7], and using the arguments in Section 2 we get the following result:

THEOREM F *Assume $f = 0$. If $u \in W^{1,2}_{\text{loc}}$ is a weak solution of* (TP)*, then $u \in W^{1,p}_{\text{loc}}$ for all $2 < p < p_0$.*

In the same way we can get an extension of Theorem E to problem (TP).

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