5. zkoušková písemka, NMAF051, ZS 2009

Každý krok krátce a správně odůvodněte.

1. Spočtěte limitu
$$\lim_{x\to 0}(1+\sin^2(x))^{\frac{1}{x-\lg(1+x)}}.$$

2. Najděte na maximálních intervalech
$$\int \frac{\sin(x)}{[\cos^2(x)+\cos(x)+1][1-\cos^2(x)]}\mathrm{d}x.$$

3. Vyšetřete průběh funkce
$$f(x)=(x+1)^{\lg(x+1)}=?$$
Studujte zejména: def. obor, spojitost, limity v krajních bodech, derivaci, její limity, monotonii, obor hodnot, druhou derivaci, konvexitu a načrtněte kvalifikovaný obrázek.

4. Najděte primitivní funkci na intervalu $(-\pi,\pi)$
$$\int |\sin(x)|\cos(\cos(x))\mathrm{d}x.$$

5. Odhadněte pomocí Lagrangeova tvaru zbytku
$$|\lg(\frac{1}{2})-T_{0,3}^{\lg(1+x)}(-\frac{1}{2})|.$$

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① $\lim\limits_{x\to 0} (1+\sin^2 x)^{\frac{1}{x-\lg(1+x)}} = \lim\limits_{x\to 0} \exp\left(\frac{1}{x-\lg(1+x)} \lg(1+\sin^2 x)\right) = L$ 1

Spočteme: $\lim\limits_{x\to 0} \frac{\lg(1+\sin^2 x)}{\sin^2 x} \frac{x^2}{x-\lg(1+x)} \cdot \frac{\sin^2 x}{x^2} \overset{AL}{=}$ 2

$\to 1$
neboť $\sin^2 x \to 0$ a
$\neq 0$ na $(-\pi,\pi)\setminus\{0\}$ 2

$= \lim\limits_{x\to 0} \frac{x^2}{x-\lg(1+x)} = \lim\limits_{x\to 0} \frac{x^2}{x-\left(x-\frac{x^2}{2}+o(x^2)\right)} = 2$ 2

a tedy $L = e^2$, protože exp je spoj. v 2. 1

② $\int \frac{-\sin x}{(\cos^2 x+\cos x+1)(\cos^2 x-1)}\,dx \overset{(P)}{=}$ 1. věta o subst. $\varphi(x)=\cos x$
$\varphi'(x) = -\sin x$
$\varphi: \mathbb{R}\to[-1,1]$ 1

Přičemž: $\int \frac{1}{(t^2+t+1)(t^2-1)}\,dt = \int \frac{At+B}{t^2+t+1} + \frac{C}{t+1} + \frac{D}{t-1}\,dt \overset{(PF)}{=}$ 1

$1 = (At+B)(t^2-1) + C(t-1)(t^2+t+1) + D(t+1)(t^2+t+1)$
$t=1:\ 1 = D\cdot 2\cdot 3 \Rightarrow D = \frac{1}{6}$
$t=-1:\ 1 = C\cdot(-2)\cdot \Rightarrow C = -\frac{1}{2}$
$t^3:\ 0 = A + C + D \Rightarrow A = -(C+D) = -\left(\frac{1}{6}-\frac{1}{2}\right) = \frac{1}{2}-\frac{1}{6} = \frac{3-1}{6} = \frac{1}{3}$
$t^0:\ 1 = -B - C + D \Rightarrow B = D - C - 1 = \frac{1}{6}+\frac{1}{2}-1 = \frac{1}{6}-\frac{1}{2} = -\frac{1}{3}$ 2

$(PF) = \int \frac{1}{3}\frac{t-1}{t^2+t+1} + \left(-\frac{1}{2}\right)\frac{1}{t+1} + \frac{1}{6}\frac{1}{t-1}\,dt = \lg\frac{\sqrt[6]{|t-1|}}{\sqrt{|t+1|}} + \frac{1}{3}\int \frac{1}{2}\frac{2t+1}{t^2+t+1} - \frac{3/2}{t^2+t+1}$

$= \lg\left(\frac{\sqrt[6]{|t-1|}}{\sqrt{|t+1|}}\cdot\sqrt[6]{t^2+t+1}\right) - \frac{1}{2}\int \frac{1}{(t+\frac{1}{2})^2+3/4}\,dt = \lg\left(\frac{\sqrt[6]{|t-1|(t^2+t+1)}}{\sqrt{|t+1|}}\right) - \frac{1}{\sqrt{3}}\cdot \operatorname{arctg}\frac{2t+1}{\sqrt{3}}$

na $(-1,1)$ a $(1,+\infty)$ a na $(-\infty,-1)$

Tedy $(P) = \lg\frac{\sqrt[6]{|\cos x-1|(\cos^2 x+\cos x+1)}}{\sqrt{|\cos x+1|}} - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\cos x+1}{\sqrt{3}}$ na $(0,\pi)+k\pi$ $\forall k\in\mathbb{Z}$, protože $\varphi: (0,\pi)+k\pi \to (-1,1)$. 1 4

③ $f(x) = (x+1)^{\lg(x+1)} = \exp(\lg^2(x+1))$

$D(f) = (-1,+\infty)$; spojitá, $\lim\limits_{x\to+\infty} f(x) = +\infty$; $\lim\limits_{x\to-1+} f(x) = +\infty$ 2

rostoucí na $[0,+\infty)$ a klesající na $(-1,0]$

$x \in D(f)$: $f'(x) = f(x)\cdot\dfrac{2\cdot\lg(x+1)}{x+1}$ 3

$f'(x) = 0 \Leftrightarrow x = 0$ a

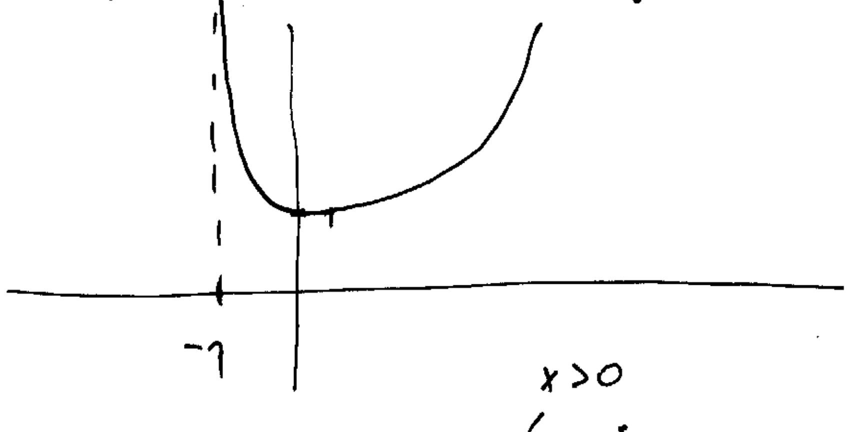
$\lim\limits_{x\to-1+} f'(x) = -\infty$; $\lim\limits_{x\to+\infty} f'(x) = \lim\limits_{x\to+\infty}(x+1)^{\lg(x+1)-1}\cdot 2\lg(x+1) = +\infty$ 1

$x\in D(f)$: $f''(x) = f(x)\cdot\dfrac{4\lg^2(x+1)}{(x+1)^2} + f(x)\dfrac{2-2\lg(x+1)}{(x+1)^2} =$

$= \dfrac{f(x)}{(x+1)^2}\left(4\lg^2(x+1) - 2\lg(x+1) + 2\right)$ 2

Existují kořeny $f''(x) \overset{?}{=} 0$? $D = 4 - 2\cdot 16 < 0 \Rightarrow$ Ne

$\Rightarrow f''(x) > 0$ na $D(f)$ $\Rightarrow$ f konvexní na $D(f)$



1

④ $\int |\sin x|\,\cancel{\cos x}\,\cos(\cos x)\,dx \overset{x>0}{=} \int \sin x\,\cancel{\cos x}\,\cos(\cos x)\,dx = (P)$

1 subst.: $\varphi(x) = \cos x$; $\varphi'(x) = -\sin x$

Spočteme: $-\int \cancel{t}\cdot\cos t\,dt = \cancel{-(t\cdot\sin t)+\int\sin t\,dt = -(t\cdot\sin t)-\cos t}$ 3

$\downarrow d$ $\downarrow i$ $\qquad -\sin t$ na $\mathbb{R}$

1 $\quad \sin t$

$\Rightarrow (P) = \begin{cases} \cancel{-\cos x\,\sin(\cos x) - \cos(\cos x)} \text{ na } (0,+\infty) = -\sin(\cos x) \\ \cancel{\cos x\,\sin(\cos x) + \cos(\cos x)} \text{ na } (-\infty,0) = \sin(\cos x) \end{cases}$

Lepeji!

$$-\sin(\cos x)$$

$$(P) := \begin{cases} -\cos x \cancel{\sin(\cos x)} - \cos(\cos x) - A & x > 0 \\ 0 \quad \sin(\cos x) & x = 0 \\ \cos x \cancel{\sin(\cos x) + \cos(\cos x)} - B & x < 0 \end{cases} \quad 2$$

kde $A = \cancel{\cos} - \cancel{1 \cdot \sin 1 - \cos x} = -\sin 1$

$B = \cancel{\sin 1 + \cos} 1 = \sin 1$

j min fce na $(-\pi, \pi)$, protože (P) i $|\sin x| \cos(\cos x)$ jsou spojité na $(-\pi, \pi)$. 1

(5) $T_{0,3}^{\lg(1+x)}\left(-\frac{1}{2}\right) = x - \frac{x^2}{2} + \frac{x^3}{3}\Big|_{x=\left(-\frac{1}{2}\right)}$ 2

$\exists \xi \in \left(-\frac{1}{2}, 0\right)$: $\lg\left(\frac{1}{2}\right) - T_{0,3}^{\lg(1+x)}\left(-\frac{1}{2}\right) = \frac{[\lg(1+x)]^{(4)}|_{x=\xi}}{4!}\left(-\frac{1}{2}\right)^4$ 2

$= R$

~~Odhad:~~ $\cancel{|R|}$

spočtu $[\lg(1+x)]^{(4)} = \left(\frac{1}{1+x}\right)^{(3)} = \left(-\frac{1}{(1+x)^2}\right)^{(2)} = \left(2 \cdot \frac{1}{(1+x)^3}\right)' =$

$$\frac{-6}{(1+x)^4}$$ 2

Tedy $|R| \leq \frac{1}{4!} \cdot |-6| \cdot \frac{1}{16} = \frac{1}{16} \cdot \frac{1}{4} = \frac{1}{64}$ 2