4. zkoušková písemka, NMAF051, ZS 2009

Každý krok krátce a správně odůvodněte.

1. Spočtěte limitu
$$\lim_{x\to 2}\left(\frac{1}{x}+\sqrt{\frac{1}{x^2}+x-2}\right)^{\operatorname{tg}(\frac{\pi x}{4})}.$$

2. Najděte na maximálních intervalech
$$\int x^3 \operatorname{arctg}(x)\mathrm{d}x.$$

3. Vyšetřete průběh funkce
$$f(x)=\arccos\left(\sqrt{\frac{x+2}{x-1}}\right).$$

   Studujte zejména: def. obor, spojitost, limity v krajních bodech, derivaci, její limity, monotonii, obor hodnot a načrtněte kvalifikovaný obrázek.

4. Všude kde existuje spočtěte derivaci funkce
$$f(x)=\begin{cases} x+\operatorname{tg}^2(x)\sin(\operatorname{cotg}(x)), & x\in(-\pi/2,\pi/2)\setminus\{0\},\\ 0, & x=0.\end{cases}$$

5. Spočtěte limitu
$$\lim_{x\to 0}\frac{\sqrt{x+1}\sin(x)+\lg(1-x)}{x^3}.$$

1) $\lim\limits_{x\to 2}\left(\frac{1}{x}+\sqrt{\frac{1}{x^2}+x-2}\right)^{\operatorname{tg}\frac{\pi x}{4}} = \lim\limits_{x\to 2}\exp\left(\operatorname{tg}\frac{\pi x}{4}\ln\left(\frac{1}{x}+\sqrt{\frac{1}{x^2}+x-2}\right)\right) = L$ 1

Spočteme: $\lim\limits_{x\to 2}\operatorname{tg}\frac{\pi x}{4}\ln\left(\frac{1}{x}+\sqrt{\frac{1}{x^2}+x-2}\right) =$

$$\lim_{x\to 2}\frac{\ln\left(\frac{1}{x}+\sqrt{\frac{1}{x^2}+x-2}\right)}{\frac{1}{x}+\sqrt{\frac{1}{x^2}+x-2}-1}\cdot\frac{\frac{1}{x}+\sqrt{\frac{1}{x^2}+x-2}-1}{}\cdot\frac{\sin\frac{\pi x}{4}}{\cos\frac{\pi x}{4}} =$$ 1

protože $\overrightarrow{\frac{1}{x}+\sqrt{\frac{1}{x^2}+x-2}}^{\,1} \to 1$ a $\frac{1}{x}+\sqrt{\frac{1}{x^2}+x-2}-1 \neq 0$ na $P(2)$

$-\left(\frac{1}{x}-1\right) \neq +\sqrt{\frac{1}{x^2}+x-2}$; $\cancel{\frac{1}{x^2}}-\frac{2}{x}+1 = \cancel{\frac{1}{x^2}}+x-2$,

$3 = x+\frac{2}{x}$; $3x = x^2+2$ ← má nejvýše 2 řešení.

Stačí vzít $P(2)$, aby je neobsahoval. 3

$$\overset{AL}{=}\lim_{x\to 2}\cancel{\frac{\left(\frac{1}{x}-1\right)^2-\left(\frac{1}{x^2}+x-2\right)}{\frac{1}{x}-1}} = \lim_{x\to 2}\frac{\frac{1}{x^2}+x-2-\left(1-\frac{1}{x}\right)^2}{\sqrt{\frac{1}{x^2}+x-2}+1-\frac{1}{x}}\cdot\frac{1}{\cos\frac{\pi x}{4}}$$

$$\overset{AL}{=}\lim_{x\to 2}\frac{x-2-1+\frac{2}{x}}{\cos\frac{\pi x}{4}} \overset{AL}{=}\lim_{x\to 2}\frac{x^2-3x+2}{\cos\frac{\pi x}{4}}\cdot\frac{1}{2}$$

$$\overset{l'H}{=}\frac{1}{2}\lim_{x\to 2}\frac{2x-3}{-\frac{\pi}{4}\sin\frac{\pi x}{4}} = \frac{1}{2}\cdot\frac{1}{-\frac{\pi}{4}} = -\frac{2}{\pi}$$ 4

a protože exp je spoj. v $-\frac{2}{\pi}$ platí $L = e^{-\frac{2}{\pi}}$ 1

2) $\int x^3\operatorname{arctg}x\,dx \overset{2}{=} \left(\frac{x^4}{4}\operatorname{arctg}x\right)-\frac{1}{4}\int\frac{x^4+x^2-x^2-1+1}{1+x^2}\,dx$

$\downarrow\int \quad \downarrow d$

$\frac{x^4}{4} \quad \frac{1}{1+x^2}$

$= \frac{x^4}{4}\operatorname{arctg}x-\frac{1}{4}\int x^2-1+\frac{1}{1+x^2}\,dx = \frac{x^4}{4}\operatorname{arctg}x$ ~~4~~ 3

$-\frac{1}{4}\left(\frac{x^3}{3}-x+\operatorname{arctg}x\right)$ na $\mathbb{R}$

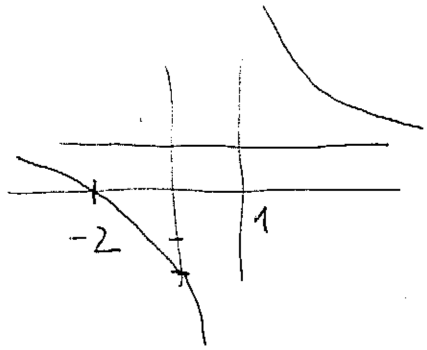
3) $f(x)=\arccos\left(\sqrt{\frac{x+2}{x-1}}\right)$

$D(f)$: $\frac{x+2}{x-1} \geq 0$

$x+2 \geq 0 \;\&\; x-1>0$, $x \geq -2$, $x>1$ $\Rightarrow$ $x>1$

$x+2 \leq 0 \;\&\; x-1<0$ $\Rightarrow$ $x \leq -2$

Obr



$\Rightarrow \sqrt{\frac{x+2}{x-1}} \in [-1,1] \Leftrightarrow x \leq -2$ 2

$\Rightarrow D(f)=(-\infty,-2]$; $f(-2)=\frac{\pi}{2}$, $\lim_{x\to-\infty} f(x)=\arccos(1)=0$ 1

3

$x\in D(f)$: $f'(x)=\frac{-1}{\sqrt{1-\frac{x+2}{x-1}}}\cdot\frac{1}{2}\sqrt{\frac{x-1}{x+2}}\cdot\frac{x-1-(x+2)}{(x-1)^2}$

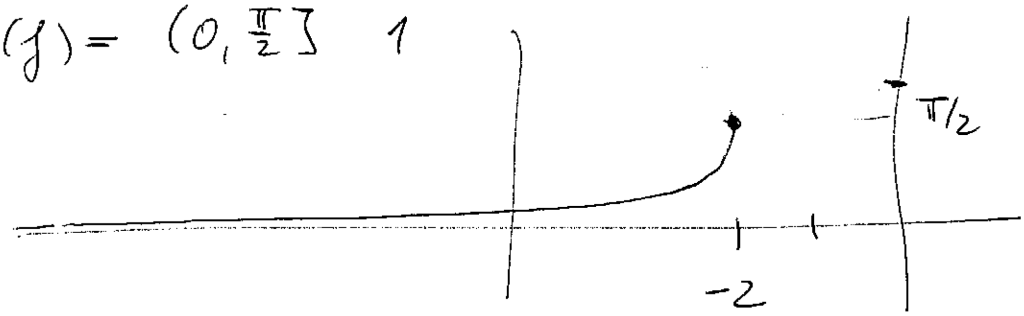
$=-\frac{1}{2}\sqrt{\frac{x-1}{-3}}\cdot\sqrt{\frac{x-1}{x+2}}\cdot\frac{-3}{(x-1)^2}>0$

$\Rightarrow f$ je rostoucí na $D(f)$ 3

$\lim_{x\to-\infty} f'(x)=0$; $\lim_{x\to-2-} f'(x)=+\infty$ 1

$\Rightarrow H(f)=(0,\frac{\pi}{2}]$ 1

Obr



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④

$$f(x) = \begin{cases} x + \operatorname{tg}^2(x)\sin(\operatorname{cotg} x) & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\} \\ 0 & x = 0 \end{cases}$$

$$\left.\begin{matrix} x \neq 0 \\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{matrix}\right\} : \quad f'(x) = 1 + 2 \cdot \operatorname{tg} x \frac{1}{\cos^2 x} \cdot \sin(\operatorname{cotg} x) + \operatorname{tg}^2 x \cos(\operatorname{cotg} x) \cdot \frac{-1}{\sin^2 x}$$

1 3

$f'(0)$ buďto limitou funkce, f spojitá v 0 nebo z definice:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - 0}{x} = 1$$

3

⑤ $$\lim_{x \to 0} \frac{\sqrt{x+1}\sin x + \lg(1-x)}{x^3} = L$$

Taylorovy polynomy: $\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{x^2}{2} + o(x^2)$ 1

$\sin x = x - \frac{x^3}{6} + o(x^3)$ 1

$\sqrt{1+x} \cdot \sin x = x + \frac{1}{2}x^2 - \frac{1}{8}x^3 - \frac{x^3}{6} + o(x^3)$ 2

$\lg(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + o(x^3)$ 1

$\sqrt{1+x} \cdot \sin x - \lg(1-x) = x^3\left(-\frac{1}{8} + \frac{1}{3}\right) + o(x^3)$

$\Rightarrow L = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$ $\quad -\frac{1}{6}$ 2