$f(x,y) = x^4 + y^4 - x^2 - 2xy - y^2$ MAF041 – DÚ 8

1) Podezřelé body: $f \in C^\infty(\mathbb{R}^2)$

$$\left.\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= 4x^3 - 2x - 2y \\ \frac{\partial f}{\partial y}(x,y) &= 4y^3 - 2x - 2y \end{aligned}\right| \begin{aligned} &= 0 \\ &= 0 \end{aligned}$$

$$\Rightarrow \quad x = y \Rightarrow \quad 4x^3 - 2x - 2x = 0 \rightarrow 4x(x^2-1) = 0$$

$$x = 0, \pm 1$$

$$(0,0) \; ; \; (1,1) \; , \; (-1,-1)$$

2) Charakterizace:

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 2 \quad ; \quad \frac{\partial^2 f}{\partial x \partial y} = -2 \quad ; \quad \frac{\partial^2 f}{\partial y^2} = 12y^2 - 2$$

v $(0,0)$: $\nabla^2 f = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$; $Q(h) = -2h_1^2 - 2h_2^2 - 4h_1h_2 = -2(h_1+h_2)^2 \le 0$

$\Rightarrow$ podezření na max. ale není protože $f(0,0) = 0 < f(x,-x)$

v $(1,1)$: $\nabla^2 f = \begin{pmatrix} 10 & -2 \\ -2 & 10 \end{pmatrix}$ pos. def dle Sylvestra $\Rightarrow$ ~~ostré~~ ostré lok. min

v $(-1,-1)$: $\nabla^2 f = \begin{pmatrix} 10 & -2 \\ -2 & 10 \end{pmatrix}$ ostré lo. min. dle Sylvestra

6) $x+y+4\cos x\cos y = f(x,y)$ na $(0,\frac{\pi}{2}] \times (0,\frac{\pi}{2})$

a) $f \in C^{\infty}(\mathbb{R}^2)$

$$\frac{\partial f}{\partial x} = 1 + 4(-\sin x)\cos y = 0$$

$$\frac{\partial f}{\partial y} = 1 + 4(\cos x)(-\sin y) = 0$$

$\Rightarrow \quad \sin x \cos y = \sin y \cos x \rightarrow \operatorname{tg} x = \operatorname{tg} y \Rightarrow x = y$

$1 + (-4)\cdot \sin x \cos x = 0 \rightarrow 1 = 2\sin 2x \rightarrow \sin 2x = \frac{1}{2}$

$\rightarrow 2x = \frac{\pi}{6}, \frac{5}{6}\pi + 2l\pi$

$x = \frac{\pi}{12}, \frac{5\pi}{12}$

podezřelé body: $\left(\frac{\pi}{12}, \frac{\pi}{12}\right)$ a $\left(\frac{5\pi}{12}, \frac{5\pi}{12}\right)$

b) Charakterizace:

$$\frac{\partial^2 f}{\partial x^2} = -4\cos x\cos y \ ; \ \frac{\partial^2 f}{\partial x \partial y} = 4\sin x \sin y \ ; \ \frac{\partial^2 f}{\partial y^2} = -4\cos x\cos y$$

v $\left(\frac{\pi}{12}, \frac{\pi}{12}\right)$: $\quad \nabla^2 f = \begin{pmatrix} -4\left(\cos\frac{\pi}{12}\right)^2 & 4\left(\sin\frac{\pi}{12}\right)^2 \\ 4\left(\sin\frac{\pi}{12}\right)^2 & -4\left(\cos\frac{\pi}{12}\right)^2 \end{pmatrix}$

$-\nabla^2 f$ je pos. def dle Sylvestra: $4\left(\cos\frac{\pi}{12}\right)^2 > 0$; $\det(-\nabla^2 f) = 16 > 0$

$\Rightarrow$ ostré lok. max.

v $\left(\frac{5\pi}{12}, \frac{5\pi}{12}\right)$: podobně ostré lok. max

7. $\sin x + \cos y + \cos(x-y) = f(x,y) \in C^{\infty}(\mathbb{R}^2)$

a) $\frac{\partial f}{\partial x} = \cos x + (-\sin(x-y)) \quad = 0$

$\frac{\partial f}{\partial y} = -\sin y + \sin(x-y) \quad = 0$

$\Rightarrow \cos x = \sin y \overset{y,x\in(0,\frac{\pi}{2})}{\Rightarrow} \sin x = \cos y$

$-\sin y + \sin(x-y) = -\sin y + \sin x \cos y - \sin y \underset{\sin y}{\cos x} \overset{!}{=} 0$

$\underset{\sin y}{\cos x} - \sin x \cos y + \sin y \underset{\sin y}{\cos x} = 0$

$\Rightarrow \sin x = \sin(\arccos(\sin y))$! $x,y \in (0,\frac{\pi}{2})$

$= \sqrt{1-\sin^2 y} = \cos y$

$\Rightarrow -\sin y + \underset{1-\sin^2 y}{\cos^2 y} - \sin^2 y = 0 \Rightarrow 1 - \sin y - 2\sin^2 y = 0$

$2\sin^2 y + \sin y - 1 = 0 \qquad \sin y = \frac{-1 \pm \sqrt{1+8}}{4} = \cancel{\frac{-1\pm\sqrt{8}}{2}} = \frac{-1\pm 3}{4} = \begin{matrix} -1 \\ \frac{1}{2} \checkmark \end{matrix}$

$\Rightarrow \sin y = \frac{1}{2} \wedge y = \frac{\pi}{6} \wedge x = \frac{\pi}{3}$

b) Charakterizace:

$\frac{\partial^2 f}{\partial x^2} = -\sin x - \cos(x-y)$; $\frac{\partial^2 f}{\partial x \partial y} = \cos(x-y)$

$\frac{\partial^2 f}{\partial y^2} = -\cos y + (-\cos(x-y))$; $* \ \frac{\pi}{3} - \frac{\pi}{6} = \frac{1}{6}\pi$

v $(\frac{\pi}{3},\frac{\pi}{6})$: $\nabla^2 f = \begin{pmatrix} -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{3} ; & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} ; & -\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} \end{pmatrix} = \frac{\sqrt{3}}{3}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

$-\nabla^2 f$ je pos. def. dle Sylvestra $\Rightarrow$ $\nabla^2 f$ je neg. def $\Rightarrow$ v $(\frac{\pi}{3},\frac{\pi}{6})$ je ostré lok. max.

8. $f(x,y) = x - 2y + \lg\sqrt{x^2+y^2} + 3\,\mathrm{arctg}\,\frac{y}{x} \qquad x \neq 0$

a) f je C^∞ na celém D_f

$$\frac{\partial f}{\partial x} = 1 + \frac{1}{\sqrt{x^2+y^2}}\cdot\frac{2x}{2\sqrt{x^2+y^2}} + 3\cdot\frac{1}{1+\frac{y^2}{x^2}}\cdot -\frac{y}{x^2} =$$

$$= \frac{x^2+y^2+x-3y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = -2 + \frac{y}{x^2+y^2} + 3\,\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1\cdot x}{x^2} = \frac{-2(x^2+y^2)+y+3x}{x^2+y^2}$$

$$\left.\begin{aligned} x^2+y^2+x-3y &= 0 \\ x^2+y^2-\frac{y}{2}-\frac{3x}{2} &= 0 \end{aligned}\right\} \Rightarrow x - 3y = -\frac{3x}{2} - \frac{y}{2}$$

$$\frac{5}{2}x = \frac{5}{2}y \Rightarrow \underline{\underline{x = y}}$$

$2x^2 + x - 3x = 0 \quad ; \quad 2x(x-1) = 0 \longrightarrow x = 0, 1$

$\Rightarrow$ krit. body $\cancel{(0,0)}$; $(1,1)$

b) Charakter: $\frac{\partial^2 f}{\partial x^2} = \frac{(2x+1)(x^2+y^2) - (x^2+y^2+x-3y)\,2x}{(x^2+y^2)^2}$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(2y-3)(x^2+y^2) - (x^2+y^2+x-3y)\,2y}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{1}{(x^2+y^2)^2}\Big((-4y+1)(x^2+y^2) - (-2(x^2+y^2)+y+3x)\,2y\Big)$$

v $(0,0)$: $\nabla^2 f$ = nemá smysl

v $(1,1)$: $\nabla^2 f = \frac{1}{4}\begin{pmatrix} \frac{3\cdot 2 - 0}{} & -2+0 \\ -1\cdot 2 - & -3\cdot 2 - 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 6 & -2 \\ -2 & -6 \end{pmatrix}$

$4Q(h) = \cancel{\tfrac{1}{4}}\left(6h_1^2 - 4h_1h_2 - 6h_2^2\right) = 6\left(h_1^2 - \frac{4}{6}h_1h_2 - h_2^2\right) = 6\left[\left(h_1 - \frac{1}{3}h_2\right)^2 - h_2^2 - \frac{h_2^2}{9}\right]$

$= 6\left[\left(h_1 - \frac{1}{3}h_2\right)^2 - \frac{10}{9}h_2^2\right] \Rightarrow$ indef volbou $(1,0), (1,3) \Rightarrow$ sedlo