DÚ 6

28) $f(x,y) = x^2+y^2-12x+16y$ na $M = \{x^2+y^2 \leq 25\}$

a) $f \in C^1(\mathbb{R}^2)$

b) M je omezená, uzavřená $\Big\} \Rightarrow$ ex. max. a min f na M

c) $\frac{\partial f}{\partial x} = 2x - 12$; $\frac{\partial f}{\partial y} = 2y + 16$

Kritické body: $x = 6$; $y = -8$ leží mimo M

$\Rightarrow$ f nabývá max. i min na ∂M

d) $x^2+y^2 = 25 \Rightarrow x = \pm\sqrt{25-y^2}$ a $-(+)$ $y \in [-5, 5]$

$g(y) = f(\sqrt{25-y^2}, y) = 25-y^2+y^2-12\sqrt{25-y^2}+16y = 25+16y-12\sqrt{25-y^2}$

$g'(y) = 16 - 12\frac{-y}{\sqrt{25-y^2}} = \frac{4}{\sqrt{25-y^2}}\left(4\sqrt{25-y^2}+3y\right)$ dopočítat!

jinak:

e) $x^2+y^2 = 25$ $x = 5\cos\varphi$, $y = 5\sin\varphi$ $\varphi \in [0, 2\pi]$

$g(\varphi) = f(5\cos\varphi, 5\sin\varphi) = 25 - 12\cdot 5\cdot\cos\varphi + 16\cdot 5\cdot\sin\varphi$

$= 25 - 60\cos\varphi + 80\sin\varphi$

$g'(\varphi) = 60\sin\varphi + 80\cos\varphi = 0$

$\cos\varphi = 0 \Rightarrow \varphi = \frac{\pi}{2} + k\pi \Rightarrow \sin\varphi = \pm 1$ současně $\sin\varphi = 0$ což nelze

$\Rightarrow$ tj. hledám $\cos\varphi \neq 0$

$\operatorname{tg}\varphi = -\frac{8}{6} = -\frac{4}{3} \Rightarrow \varphi \in \{\underbrace{\operatorname{arctg}(-\frac{4}{3})}_{\varphi_1}, \underbrace{\operatorname{arctg}(-\frac{4}{3})+\pi}_{\varphi_2}\}$

V krajních bodech extrém není $\Rightarrow$

$\min_M f = f(5\cos\varphi_1, 5\sin\varphi_1) = f(3,-4) = 25-36-64 = -75$

$\max_M f = f(5\cos\varphi_2, 5\sin\varphi_2) = f(-3,4) = 125$

29) $f(x,y,z) = x+y+z$, spoj. $(\in C^\infty)$

$M = \{ x^2+y^2 \le z \le 1 \}$ je kpt. ⇒ $\Big\}$ ⇒ ex. min a max.

Ale nelze použít přímo $\frac{\partial f}{\partial x} = 1$.

1) $z = 1$ $\quad g(x,y) = x+y+1$ hled. max. nelze použít $x^2+y^2 \le 1$.

$x^2+y^2 = 1$

~~$g(x) = x \pm \sqrt{1-x^2}$ (pouze 1. část)~~

~~$g'(x) = 1 + \frac{-x}{\sqrt{1-x^2}}$~~

$x = \cos\varphi$, $y = \sin\varphi$

$g(\varphi) = \cos\varphi + \sin\varphi + 1$;

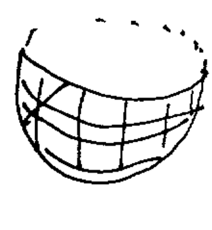
$g'(\varphi) = \cos\varphi - \sin\varphi$

⇒ krit. bod $\quad \varphi = \frac{\pi}{4} + k\pi \quad k \in \mathbb{N}$

Kandidáti: $f\left(\frac{\sqrt2}{2}, \frac{\sqrt2}{2}, 1\right) = 1+\sqrt2$

$f\left(-\frac{\sqrt2}{2}, -\frac{\sqrt2}{2}, 1\right) = 1-\sqrt2$

2) Zbývá vyšetřit: $\quad x^2+y^2 = \cancel{z} \quad z \in [0,1]$



$g(x,y) = x+y+x^2+y^2$

$$\left.\begin{aligned} \frac{\partial g}{\partial x} &= 1+2x \\ \frac{\partial g}{\partial y} &= 1+2y \end{aligned}\right\} \Rightarrow \text{krit. bod } x = -\tfrac12 = y \ ; \ z = \tfrac12$$

$f\left(-\frac12, -\frac12, \frac12\right) = -\frac12$

Krajní body $z=0 \Rightarrow x=0=y \Rightarrow f(0,0,0) = 0$

$z = 1$ bylo vyšetřeno v 1)

$\Rightarrow \max_M f = 1+\sqrt2$ v $\left(\frac{\sqrt2}{2}, \frac{\sqrt2}{2}, 1\right)$ a $\min_M f = -\frac12$ v $\left(-\frac12, -\frac12, \frac12\right)$

30) $V = abc > 0$

$$S = 2(ab + bc + ac)$$

$$f(a,b,c) = 2(ab+bc+ac) \quad \text{na } M = \{abc = V;\ a \geq 0,\ b \geq 0,\ c \geq 0\}$$

Pozor M není kompakt! Není omezená!!

1) ~~Když~~ v M: $a>0, b>0, c>0$ vyjádříme $a = \frac{V}{bc}$

$$g(b,c) = f\left(\frac{V}{bc}, b, c\right) = 2\left(\frac{V}{c} + bc + \frac{V}{b}\right)$$

na $\cancel{\quad}$ $\{b > 0, c > 0\}$

$g(1,1) = 2V + 1$

pokud $c < \frac{1}{4+\frac{1}{V}} \Rightarrow g > 2(4V+1)$ podobně s b

$\Rightarrow$ "c" a "b" $\geq \frac{1}{4+\frac{1}{V}}$ ← budeme psát.

$b > \left(4+\frac{1}{V}\right)\cdot(4V+1) \Rightarrow g > 2(4V+1)$ podobně c

$\Rightarrow g$ nabývá minima uvnitř $N := \left[\frac{V}{4V+1}, 4V+1\right]^2$.

$$\frac{\partial g}{\partial b} = 2\cdot\left(c - \frac{V}{b^2}\right)$$

$$\frac{\partial g}{\partial c} = 2\left(\frac{V}{c^2} - b\right)$$

Krit. body: $c = \frac{V}{b^2} \Rightarrow b = \frac{V}{\left(\frac{V}{b^2}\right)^2}$

$b = \frac{b^4}{V} \Rightarrow b^3 = V \Rightarrow b = \sqrt[3]{V}$

$\Rightarrow c = \sqrt[3]{V}$

Jediný krit. bod $\Rightarrow \min\limits_{M} f = 2\cdot 3\cdot V^{2/3}$ v $(\sqrt[3]{V}, \sqrt[3]{V}, \sqrt[3]{V})$.