DÚ – 11

1) Transformujte $\dfrac{\partial^2 u}{\partial x \partial y}$ do $\begin{array}{l} x = s - t \\ y = s + t \end{array}$

2) Transformujte Δ v $\mathbb{R}^3$ do sférických souřadnic (úmyslně)
sférické = (polární)^2

3) Transformujte Δ v $\mathbb{R}^n$ na $\{(x_1, \dots, x_n) ;\ x_n > \varphi(x_1, \dots, x_{n-1})\}$
na $\{(y_1, \dots, y_n) ;\ y_n > 0\}$

Návod: zobrazení: $y_i = x_i \qquad i = 1, \dots, n-1$
$y_n = x_n - \varphi(x_1, \dots, x_{n-1})$

ad 1) $(x,y) \longrightarrow (s,t)$ je prostá na $\mathbb{R}^2 \to \mathbb{R}^2$

$$s,t \longrightarrow x,y \xrightarrow{u} \mathbb{R}$$

$v(s,t) := u(x,y)$ ~~Podle věty o IF bez újmy~~?

$$s = \frac{x+y}{2} \ ; \quad t = \frac{y-x}{2} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial s} \cdot \frac{1}{2} + \frac{\partial v}{\partial t} \cdot \left(-\frac{1}{2}\right)$$

$$\Rightarrow \quad \frac{\partial u}{\partial x \partial y}(x,y) = \frac{\partial^2 v}{\partial s^2} \cdot \frac{1}{4} + \frac{\partial^2 v}{\partial s \partial t} \cdot \frac{1}{4} + \frac{\partial^2 v}{\partial t \partial s}\left(-\frac{1}{4}\right) + \frac{\partial^2 v}{\partial t^2}\left(-\frac{1}{4}\right)$$

$$= \frac{\partial^2 v}{\partial s^2}(s,t) - \frac{\partial^2 v}{\partial t^2}(s,t)$$

$\Rightarrow$ Vlnovou rovnici v $\mathbb{R}^2$ lze převést na tvar $\dfrac{\partial u}{\partial x \partial y} = f$ kterou lze řešit přímou integrací!

Vlnová rovnice:

$$\frac{\partial^2 v}{\partial s^2}(s,t) - \frac{\partial^2 v}{\partial t^2}(s,t) = 0$$

$$v(s,0) = v_0(s)$$

$$\frac{\partial v}{\partial t}(s,0) = v_1(s)$$

Lze předchozí transformací převést na $\frac{\partial^2 u}{\partial x \partial y}(x,y) = 0 \Rightarrow$

$$\frac{\partial u}{\partial x}(x,y) = C(x) \Rightarrow u(x,y) = D(y) + \underbrace{\int_0^x C(x)\,dx}_{\text{označ } E(x)}$$

$$\Rightarrow v(s,t) = D(s+t) + E(s-t)$$

Počáteční podmínky:

$$v(s,0) = D(s) + E(s) = v_0(s)$$

$$\frac{\partial v}{\partial t}(s,0) = D'(s) - E'(s) = v_1(s)$$

$$\Rightarrow D(s) - E(s) = \int_0^s v_1(\alpha)\,d\alpha$$

$$\Rightarrow D(s) = \frac{1}{2}\left(v_0(s) + \int_0^s v_1(\alpha)\,d\alpha\right)$$

$$E(s) = \frac{1}{2}\left(v_0(s) - \int_0^s v_1(\alpha)\,d\alpha\right)$$

$\Rightarrow$ Řešení Cauchyovy úlohy pro vlnovou rovnici má tvar:

$$v(s,t) = \frac{1}{2}\left(v_0(s+t) + v_0(s-t) + \int_0^{s+t} v_1(\alpha)\,d\alpha - \int_0^{s-t} v_1(\alpha)\,d\alpha\right)$$

$$\forall s \in \mathbb{R},\ t \in \mathbb{R}$$

Pozn! nevyžadovali jsme žádné předpoklady, nemáme pojem řešení, výpočet byl tedy pouze formální!

Ad 2) sférické souřadnice: $x = r\cdot\cos\varphi\cos\psi$ $\quad\varphi\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$y = r\cdot\cos\varphi\sin\psi$ $\quad\psi\in(-\pi,\pi)$

$z = r\cdot\sin\varphi$

Je prosté zobrazení na prostor

$(x,y,z)\in \cancel{\{(x,y,z);\ y\le 0,}\ \mathbb{R}^3\setminus\{(x,0,z);\ z\in\mathbb{R},\ x\le 0\}$

$(r,\varphi,\psi)\in(0,+\infty)\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times(-\pi,\pi)$

$$(r,\varphi,\psi)\longrightarrow(u,v,\psi)\longrightarrow(x,y,z)$$

$u = r\cos\varphi$ $\qquad x = u\cos\psi$

$v = r\sin\varphi$ $\qquad y = u\sin\psi$

$\psi = \psi$ $\qquad z = v$

Jak vypadá trans. Δ do (u,v,ψ)? označme $f(x,y,z) = g(u,v,\psi)$

$$\Delta f(x,y,z) \underset{\text{cv.}}{=} \left(\frac{\partial^2 g}{\partial u^2}+\frac{1}{u^2}\cdot\frac{\partial^2 g}{\partial\psi^2}+\frac{1}{u}\frac{\partial g}{\partial u}\right)(u,v,\psi)+\frac{\partial^2 g}{\partial v^2}(u,v,\psi)$$

$$=\frac{\partial^2 g}{\partial u^2}+\frac{\partial^2 g}{\partial v^2}+\frac{1}{u^2}\frac{\partial^2 g}{\partial\psi^2}+\frac{1}{u}\frac{\partial g}{\partial u}$$

Označ. $h(r,\varphi,\psi) = g(u,v,\psi)$ a cv. plyne:

$$\frac{\partial^2 g}{\partial u^2}+\frac{\partial^2 g}{\partial v^2} = \frac{\partial^2 h}{\partial r^2}+\frac{\partial^2 h}{\partial\varphi^2}\cdot\frac{1}{r^2}+\frac{1}{r}\cdot\frac{\partial h}{\partial r}$$

Zbývá $\frac{\partial g}{\partial u}$: a cv. $\frac{\partial g}{\partial u} = \frac{\partial h}{\partial r}\cdot\frac{\partial r}{\partial u}+\frac{\partial h}{\partial\varphi}\cdot\frac{\partial\varphi}{\partial u} =$

$$\frac{\partial h}{\partial r}\cdot\cos\varphi+\frac{\partial h}{\partial\varphi}\cdot\left(-\frac{\sin\varphi}{r}\right)$$

derivace ~~[illegible]~~ proměnné! dle φ,ψ

$$\Rightarrow\ \Delta f(x,y,z) = \frac{\partial^2 h}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 h}{\partial\varphi^2}+\frac{1}{r}\frac{\partial h}{\partial r}+\frac{1}{r^2\cos^2\varphi}\frac{\partial^2 h}{\partial\psi^2}+$$

$$\frac{1}{r\cos\varphi}\left(\frac{\partial h}{\partial r}\cos\varphi-\frac{\partial h}{\partial\varphi}\frac{\sin\varphi}{r}\right) = \frac{\partial^2 h}{\partial r^2}+\frac{2}{r}\frac{\partial h}{\partial r}+\frac{1}{r^2}\left(\frac{\partial^2 h}{\partial\varphi^2}+\frac{\partial^2 h}{\cos^2\varphi\,\partial\psi^2}+\dots\right)$$

Poznámka: Pro radiálně symetrickou f platí:

$$f(x,y,z) = g(r) \text{ a } \Delta f = \frac{\partial^2 g}{\partial r^2} + \frac{2}{r}\frac{\partial g}{\partial r} \quad !$$

ad 3) Zobrazení lze zapsat takto:

$$x_i = y_i$$

$$x_n = y_n + \varphi(\underbrace{y_1, \dots, y_{n-1}}_{n-1})$$

a lze vypočítat pomocí

$$\det \begin{pmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ \frac{\partial \varphi}{\partial y_1} & \cdots & \frac{\partial \varphi}{\partial y_{n-1}} & 1 \end{pmatrix} \neq 0 \quad \checkmark$$

Definuji $u(x_1, \dots, x_n) =: v(y_1, \dots, y_n)$
Je vidět:

$$\cancel{\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i}} \quad \text{ale } i \in \{1, \dots, n-1\}$$

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i} + \frac{\partial v}{\partial y_n} \cdot \frac{\partial y_n}{\partial x_i} = \frac{\partial v}{\partial y_i} + \frac{\partial v}{\partial y_n}\left(-\frac{\partial \varphi}{\partial x_i}\right)$$

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 v}{\partial y_i^2} + 2\frac{\partial^2 v}{\partial y_i \partial y_n}\left(-\frac{\partial \varphi}{\partial x_i}\right) + \frac{\partial^2 v}{\partial y_n^2} \cdot \left(\frac{\partial \varphi}{\partial x_i}\right)^2 + \frac{\partial v}{\partial y_n}\left(-\frac{\partial^2 \varphi}{\partial x_i^2}\right)$$

$$\frac{\partial u}{\partial x_n} = \frac{\partial v}{\partial y_n} \; ; \quad \frac{\partial^2 u}{\partial x_n^2} = \frac{\partial^2 v}{\partial y_n^2} \quad \Rightarrow$$

$$\Delta_x u(x) = \Delta_y v(y) - 2\left(\nabla_y \frac{\partial v}{\partial y_n} \cdot \nabla_x \varphi\right) + \left(-\frac{\partial v}{\partial y_n}\right) |\nabla \varphi|^2$$

Vyjádřím ještě $\frac{\partial}{\partial x_i}$ pomocí $\frac{\partial}{\partial y_i}$

Platí $i \in \{1, \dots, n-1\}$

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i} - \frac{\partial \varphi}{\partial x_i} \cdot \frac{\partial}{\partial y_n} \, , \quad \frac{\partial}{\partial x_n} = \frac{\partial}{\partial y_n}$$

Tedy: $\cancel{\Delta u = \operatorname{div}_x(\nabla_x u) = \operatorname{div}_x(\nabla_y}$

$$\nabla_x u = A(y)\cdot\nabla_y v \qquad A(y) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -\frac{\partial\varphi}{\partial x_i} \\ 0 & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & 0 & -\frac{\partial\varphi}{\partial x_{n-1}} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Zároveň $\cancel{\operatorname{div}_x(f_1, \dots, f_n) = \operatorname{div}_y(}$

$\cancel{\operatorname{div}_x(f_1, \dots, f_n) = \operatorname{div}_y\left(\vec{g}_1, \dots, \vec{g}_{n-1}, \left(1 - \sum_{i=1}^{n-1}\frac{\partial\varphi}{\partial x_i}\right)\vec{g}_n\right)}$

$\cancel{= \operatorname{div}_y(}$

Zároveň pro $g, f : \mathbb{R}^n \to \mathbb{R}^n$; $f(x_1, \dots, x_n) = g(y_1, \dots, y_n)$

$$\operatorname{div}_x f = \operatorname{div}_y\left(g_1, \dots, \left(1 - \sum_{i=1}^{n-1}\frac{\partial\varphi}{\partial x_i}\right)g_n\right) =$$

$$= \operatorname{div}_y\big((A(y)_i\, g)\big) \Rightarrow$$

$$\Delta_x u = \operatorname{div}_x(\nabla_x u) = \operatorname{div}_y\big((A(y), \nabla_x v(y)\big) =$$

$$\operatorname{div}_y\big((A(y)_i \cdot A(y)\cdot\nabla_y v(y)\big) = \operatorname{div}_y\big((A(y))^2\cdot\nabla_y v\big)$$

$$\big((A(y))^2\big)_{ij} = \sum_{k=1}^{n} A_{ik} A_{kj}$$