DÚ 10 – MAF091

Kap II: 113/53 $\quad F(x,y,z) = xyz - x - y - z$

Spočteme: $\frac{\partial F}{\partial z}(x,y,z) = xy - 1 \Rightarrow$ Pokud $xy \neq 1$ je $\frac{\partial F}{\partial z}(x,y,z) \neq 0$

Předpokládejme, že $F(x_0,y_0,z_0) = 0$ a $x_0 y_0 \neq 1$. Pak jsou splněny předpoklady VOIP: $F \in C^\infty \Rightarrow$ na okolí bodu (x_0,y_0,z_0) lze z vyjádřit jako fci x,y; $z(x,y) \in C^2(\mathbb{R}^2)$.

Počítejme derivace z.

$$F(x,y,z(x,y)) = 0 \Rightarrow \frac{\partial F}{\partial x} = yz - 1 + (xy-1)\frac{\partial z}{\partial x}(x,y) = 0$$

$$\Rightarrow \frac{\partial z}{\partial x}(x,y) = \frac{1 - y\,z(x,y)}{xy-1}, \text{ podobně: } \frac{\partial z}{\partial y}(x,y) = \frac{1 - x\,z(x,y)}{xy-1}$$

dále:
$$\frac{\partial^2 z}{\partial x^2}(x_0,y_0) = \frac{-y\frac{\partial z}{\partial x}\cdot(xy-1) - (1-y\,z(x,y))\cdot y}{(xy-1)^2}\bigg|_{(x_0,y_0)} = \frac{(1-y_0z_0)(-2y_0)}{(x_0y_0-1)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y}(x_0,y_0) = \frac{\left(-z - y\frac{\partial z}{\partial y}\right)(xy-1) - x(1-yz)}{(xy-1)^2}\bigg|_{(x_0,y_0)} = \frac{\left(-z - y\frac{1-xz}{xy-1}\right)(xy-1) - x(1-yz)}{(xy-1)^2}\bigg|_{(x_0,y_0)}$$

$$= \frac{1}{(xy-1)^2}\left(-\cancel{xyz} + z - y + \cancel{xyz} - x + xyz\right)\bigg|_{(x_0,y_0)} = \frac{x_0y_0z_0 + z_0 - x_0 - y_0}{(x_0y_0-1)^2}$$

Konečně: $$\frac{\partial^2 z}{\partial y^2}(x_0,y_0) = \frac{-x\frac{\partial z}{\partial y}(xy-1) - (1-xz)x}{(xy-1)^2}\bigg|_{(x_0,y_0)} = \frac{-2x_0(1-x_0z_0)}{(x_0y_0-1)^2}$$

1. Diferenciál: $$dz = \frac{1-yz}{xy-1}dx + \frac{1-xz}{yx-1}dy$$

2. dif: $$d^2z = \frac{-2}{(xy-1)^2}\left((1-yz)y(dx)^2 + (xy - z - xyz)\,dx\,dy + x(1-xz)(dy)^2\right)$$

Kap II 113/58 :

$du = ?$ $\quad u^3 - 3(x+y)u^2 + z^3 = 0$

Def: $F(x,y,z,u) = u^3 - 3(x+y)u^2 + z^3$

1. ať v (x_0, y_0, z_0, u_0) platí $F(x_0, y_0, z_0, u_0) = 0$. Předpoklad!
2. $F \in C^\infty(\mathbb{R}^4)$
3. $\frac{\partial F}{\partial u} = 3u^2 - 6u(x+y)$ Tedy pokud $u_0 \neq 0$ a

$x_0 + y_0 \neq \frac{1}{2}$ lze vyjádřit u v okolí (x_0, y_0, z_0).
a i je to derivace

Pak:
$$\frac{\partial u}{\partial x} = \frac{+3u^2}{3u(u-2(x+y))} = \frac{u}{u-2(x+y)}$$
$$\frac{\partial u}{\partial y} = \frac{3u^2}{3u(u-2(x+y))} = \frac{u}{u-2(x+y)}$$
$$\frac{\partial u}{\partial z} = \frac{-3z^2}{3u(u-2(x+y))}$$

$$\Rightarrow du = \frac{u}{u-2(x+y)}(dx+dy) - \frac{z^2}{u(u-2(x+y))}dz$$

! v okolí bodu (x_0, y_0, z_0) daném nahoře.

56) $x = f(y,z)$; $y = g(x,z)$ a $z = h(x,y)$ a platí:
$$F(f(y,z),y,z) = F(x,g(x,z),z) = F(x,y,h(x,y)) = 0$$
Předpokládejme, že ~~...~~ rovnost nahoře platí v nějakém x_0, y_0, z_0 a navíc v tomto bodě platí $\frac{\partial F}{\partial x} \neq 0$; $\frac{\partial F}{\partial y} \neq 0$, $\frac{\partial F}{\partial z} \neq 0$ a $F \in C^1$ na okolí tohoto bodu. Pak VOIF $\Rightarrow$ ex. f, g, h na vhodném okolí.

Spočtěme f_y:
$$f_y = -\frac{\partial F}{\partial y} \Big/ \frac{\partial F}{\partial x}$$

Podobně:

$$g_z = -\frac{\partial F}{\partial z} \Big/ \frac{\partial F}{\partial y} \quad ; \quad h_x = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z}$$

Dohromady tedy $f_y \, g_z \, h_x = -1$. (Pozor na předpoklady!)

11) $z(x,y)$ je zadána vztahem $F(x + z y^{-1}, y + z x^{-1}) = 0$

na okolí bodu (x_0, y_0)

předpokl.: 1) $F(x_0 + z_0/y_0, y_0 + z_0/x_0) = 0$ a $z_0 = z(x_0, y_0)$

2) $F \in C^1$ na okolí (x_0, y_0, z_0) a $y_0 \neq 0$, $x_0 \neq 0$

3) $\frac{d}{dz}\left(F(x + z/y, y + z/x)\right) = \partial_1 F(x + z/y, y + z/x) \cdot \left(+\frac{1}{y}\right) +$

$+ \partial_2 F(x + z/y, y + z/x) \cdot \frac{1}{x}$. Budeme předpokládat, že

$\frac{d}{dz}\left(F(x + \frac{z}{y}, y + z/x)\right) \neq 0$.

Navíc, protože F je C^1 lze spočítat derivace na okolí bodu (x_0, y_0)

$$\frac{\partial z}{\partial x} = -\left(\partial_1 F + \partial_2 F\left(-\frac{1}{x^2}\right)\right) \Big/ \left(\partial_1 F/y + \partial_2 F/x\right)$$

$$\frac{\partial z}{\partial y} = -\left(\partial_1 F(-1/y^2) + \partial_2 F\right) \Big/ \left(\partial_1 F/y + \partial_2 F/x\right)$$

Dosadíme do $x z_x + y z_y = z - xy \Rightarrow$

$$\left(-x \cdot \partial_1 F + \partial_2 F/x + \partial_1 F/y - y \partial_2 F\right) \Big/ \left(\partial_1 F/y + \partial_2 F/x\right) =$$

$$\cancel{-(\partial_1 F + \partial_2 F)\left(x + y - \frac{1}{x} - \frac{1}{y}\right)} \Big/ \left(\partial_1 F/y + \partial_2 F/x\right)$$

Tedy:

$$\cancel{-(\partial_1 F + \partial_2 F)\left(x + y - \frac{1}{x} - \frac{1}{y}\right)} = \frac{z}{y} \partial_1 F + \frac{z}{x} \partial_2 F - x \partial_1 F - y \partial_2 F$$

$$-x \partial_1 F - y \partial_2 F + z \partial_1 F/y + \partial_2 F/x \, z$$

OK.