MAF031

DÚ: 4a polární, pozor na meze

12 5e válcové

5g sférické

10 válcové

4a): $M = \{(x^2+y^2)^2 < 2a^2(x^2-y^2)\}$ protože $N = \{ \ldots > \ldots \}$ není omezená! např. $(x_A, y_A) = (1,-1)$

Které r, φ v pol. souřadnicích odpovídají M?

$$r^4 < 2a^2(r^2\cos^2\varphi - r^2\sin^2\varphi) = 2a^2r^2\cos 2\varphi$$

$r^2 < 2a^2\cos 2\varphi$ ← musí být kladné $\Rightarrow 2\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \varphi \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

Fubini

$$\Rightarrow \mu_2(M) = \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{2\cos 2\varphi}} r \, dr \, d\varphi = \int_{-\pi/4}^{\pi/4} \frac{1}{2}a^2 \cdot 2\cos 2\varphi \, d\varphi = \left[\frac{\sin 2\varphi}{2}\right]_{-\pi/4}^{\pi/4} \cdot a^2$$

$$= \frac{a^2}{2}$$

5e) $M = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1; \ \frac{x^2}{a^2} + \frac{y^2}{b^2} \le \frac{z^2}{c^2} \right\}$

Válcové:

$x = ra\cos\varphi$

$y = rb\sin\varphi$

$z = cz$

$$\text{Jac} = \det\begin{pmatrix} a\cos\varphi & -ra\sin\varphi & 0 \\ a\sin\varphi & rb\cos\varphi & 0 \\ 0 & 0 & c \end{pmatrix} = abcr$$

$$\mu_3(M) = \int_N r = 2\int_{-\pi}^{\pi}\left(\int_0^1\left(\int_r^1 r \, dz\right)dr\right)d\varphi = 2\int_{-\pi}^{\pi}\int_0^1 r(1-r)\,dr\,d\varphi =$$

$N = \{ r^2 + z^2 < 1, \ r^2 < z^2, \ \varphi \in (-\pi, \pi) \}$

$$4\pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{2}{3}\pi$$

5g: $M = \{ (x^2+y^2+z^2)^3 < 3xyz \}$ Proč? $(x=0, y=z)$

$N = \{ r^6 < 3r^3 \cos\varphi \cos^2\psi \sin\varphi \sin\psi \}$

$\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$; $\varphi \in (-\pi, \pi)$

$\cos\varphi \sin\varphi = \frac{1}{2} \sin 2\varphi > 0$ pokud $\varphi \in (0, \pi/2) \cup (-\pi, -\frac{\pi}{2})$

$\sin\psi > 0$ pokud $\psi \in (0, \pi/2)$

Tedy v N musí: $\varphi \in (0, \pi/2) \cup (-\pi, -\pi/2)$ a $\psi \in (0, \frac{\pi}{2})$

nebo $\varphi \in (\frac{\pi}{2}, \pi) \cup (-\frac{\pi}{2}, 0)$ a $\psi \in (-\frac{\pi}{2}, 0)$

ale ty části jsou stejné.

$$\mu_3(M) = \int_N r\cos\psi \overset{\text{Pouze 1 část.}}{=} \int_0^{\pi/2} \left(\int_0^{\pi/2} \left(\int_0^{\sqrt[3]{3\cos\varphi\cos^2\psi\sin\varphi\sin\psi}} r^2 \cos\psi \, dr \right) d\psi \right) d\varphi =$$

$$= \int_0^{\pi/2} \left(\int_0^{\pi/2} \frac{1}{3} \cdot 3 \cos\varphi \sin\varphi \cos^3\psi \sin\psi \, d\psi \right) d\varphi =$$

$$\left[\frac{1}{4} \cos 2\varphi \right]_0^{\pi/2} \left[-\frac{\cos^4\psi}{4} \right]_0^{\pi/2} = \frac{1}{2}\frac{1}{8} (-2) \cdot (-1) = \frac{1}{8}$$

Symetrie (ověřte) $\mu_3(M) = \frac{1}{2}$.

10) $V = \{ z \in (0, l);\ x^2+y^2 < R^2 \}$

$$m = \int_V \rho \underset{\text{Fubini}}{=} \int_{\{x^2+y^2<R^2\}} \left(\int_0^l \rho_0 \exp\left(-\frac{\rho_0 g}{p_0} z\right) dz \right) dx\,dy =$$

$$= \int_{\{x^2+y^2<R^2\}} \left[\frac{\rho_0 \cdot \exp\left(-\frac{\rho_0 g}{p_0} z\right)}{-\frac{\rho_0 g}{p_0}} \right]_{0=z}^{l} dx\,dy =$$

$$= \int_{\{x^2+y^2<R^2\}} \frac{p_0}{g}\left(1 - e^{-\frac{\rho_0}{p_0} \cdot lg}\right) dx\,dy \underset{\text{obsah kruh}}{=} \frac{\pi R^2 p_0}{g}\left(1 - e^{-\frac{\rho_0}{p_0} lg}\right)$$