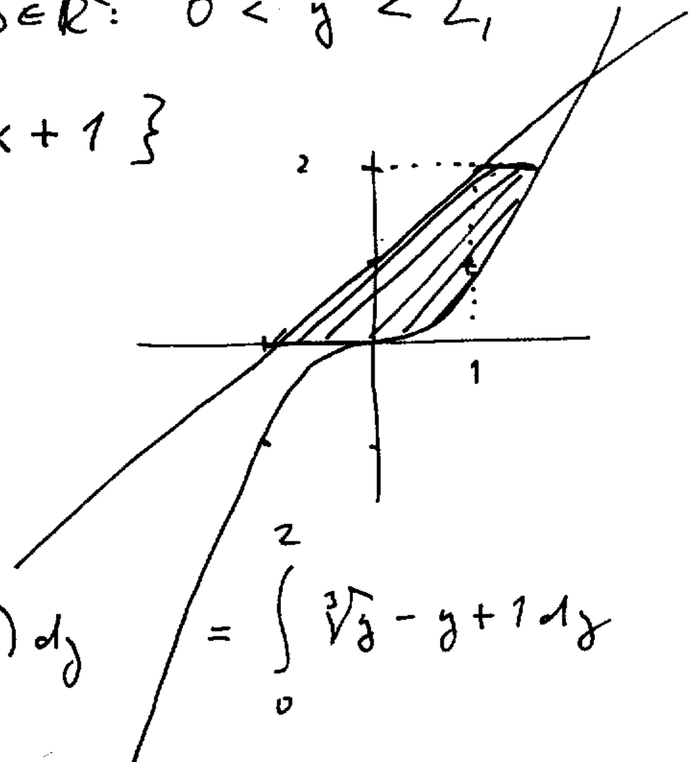
DÚ: 11   MAF034

1) Spočtěte plochu: $M = \{(x,y) \in \mathbb{R}^2 : 0 < y < 2,$

$x^3 < y < x + 1\}$



$x^3 = x + 1$

$x^3 - x = +1$?

$$|M| = \int_0^2 \Big(\int_{y-1}^{\sqrt[3]{y}} 1 \, dx \Big) dy = \int_0^2 \sqrt[3]{y} - y + 1 \, dy$$

$$= \Big[\frac{y^{4/3}}{\frac{4}{3}} - \frac{y^2}{2} + y \Big]_0^2 = \frac{3 \cdot \sqrt[3]{16}}{4} - 2 + 2 = \frac{3}{2}\sqrt[3]{2}$$

$M = \{(x,y) \in \mathbb{R}^2 : 0 < y < 2;\ x^3 < y < x\}$

$$|M| = \int_0^1 \int_y^{\sqrt[3]{y}} 1 \, dx \, dy = \int_0^1 \sqrt[3]{y} - y \, dy = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

2) Určete objem tělesa: ohraničeného plochami:

$x + y + z = a;\ x^2 + y^2 = R^2;\ x = 0;\ y = 0;\ z = 0;\ a \geq R\sqrt{2} > 0$

Omezená je část prostoru: $M := \{x^2 + y^2 \leq R^2,\ x + y + z < a;\ x > 0, y > 0, z > 0\}$

spočteme průmět do roviny (x,y): $x > 0, y > 0, x^2 + y^2 \leq R^2$;

a $x + y < a - z$ a $x > 0, y > 0$ dostáváme: $x + y < a$

z pak patří do: $z \in (0, a - (x+y))$

Značí: $N \subset \mathbb{R}^2 : N = \{x > 0, y > 0, x^2 + y^2 < R^2;\ x + y < a\}$

$$\Rightarrow |M| = \int_N \Big(\int_0^{a-(x+y)} dz \Big) d(x,y) = \int_N a - x - y \, (dx\,dy)$$ a polárně na N

$$N = \{x>0, y>0 : x^2+y^2 < R^2, x+y < a\}$$

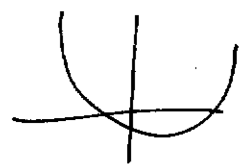
$$x < \sqrt{R^2-y^2} \quad \& \quad x < a-y$$

Kdy bude [illegible] nerovnost [illegible]?

$$\sqrt{R^2-y^2} \overset{?}{<} a-y$$

$$R^2-y^2 < a^2-2ay+y^2$$

$$0 < 2y^2-2ay+a^2-R^2$$



$$y_{1,2} = \frac{2a \pm \sqrt{4a^2-8(a^2-R^2)}}{4} = \frac{2a \pm 2\sqrt{2R^2-a^2}}{4} = \frac{a}{2} \pm \sqrt{2R^2-a^2}$$

Pozn: $2R^2-a^2 < 0 \Rightarrow$ vždy: $\sqrt{R^2-y^2} < a-y$

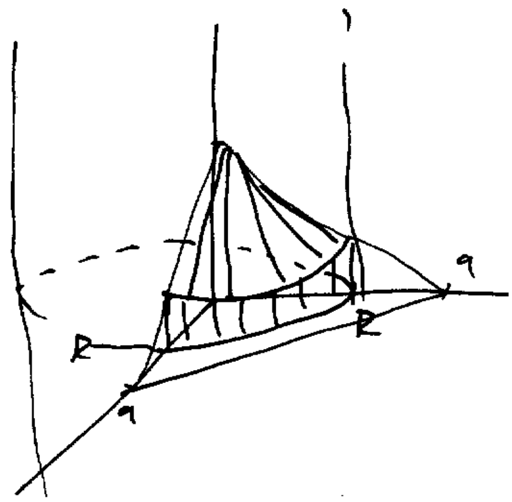
$$\Rightarrow N = \{x>0, y>0, x^2+y^2 < R^2\}$$

$$\Rightarrow |M| = \int_0^R \int_0^{\sqrt{R^2-x^2}} (a-x-y)\,dy\,dx = \int_0^R (a-x)\sqrt{R^2-x^2} - \frac{R^2-x^2}{2}\,dx$$

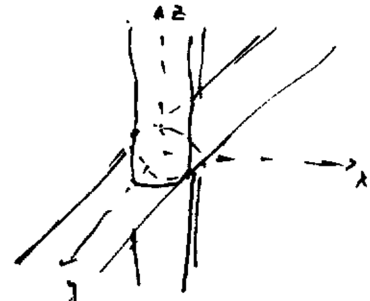
$$= \int_0^R a\cdot\sqrt{R^2-x^2}\,dx - \int_0^R x\sqrt{R^2-x^2}\,dx - \frac{1}{2}\left[R^2x - \frac{x^3}{3}\right]_0^R =$$

$$R^3\left(\frac{1}{6}-\frac{1}{2}-\frac{1}{3}\right) + \frac{1}{2}aR^2\frac{\pi}{2} = R^2\left(\frac{\pi a}{4} - \frac{2}{3}R\right)$$

Určitě by to [illegible]? Obr M.



asi ne!



DÚ: Průnik dvou válců

$$M = \{ x^2+y^2 < R^2 ;\ x^2+z^2 < R^2 \}$$

Jak vypadá průmět do x? $x \in (-R,R)$

Jak vypadá pro pevné x řez? $\{y^2 < R^2-x^2 ;\ z^2 < R^2-x^2\} =$

$$= \{ |y| < \sqrt{R^2-x^2} ;\ |z| < \sqrt{R^2-x^2} \} =: N_x$$

$$|M| = \int_0^R \Big(\int_{N_x} dz\,dy \Big)\, dx = \int_0^R 4(R^2-x^2)\,dx = 4\Big[R^2x - \frac{x^3}{3}\Big]_0^R =$$

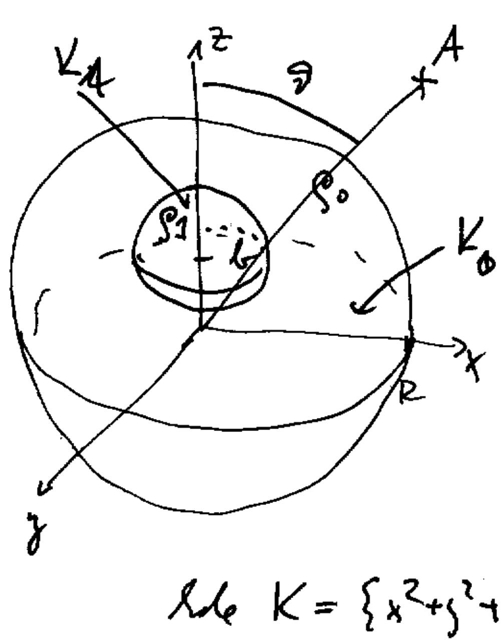
$$= 4R^3\Big(1-\frac{1}{3}\Big) = \frac{8}{3}R^3$$

Jde to i jinak – složitěji. Např: průmět do (x,y) je $\{x^2+y^2<R^2\}$

$z \in (-\sqrt{R^2-x^2}, \sqrt{R^2-x^2})$ a tedy:

$$|M| = \int_{\{x^2+y^2<R^2\}} \Big(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dz \Big)\, dx\,dy \quad \ldots$$

Kap III 54/35:



$$N = \int_{K_0} \frac{\varrho_0}{|A-X|}\,dx\,dy\,dz + \int_{K_1} \frac{\varrho_1}{|A-X|}\,dx\,dy\,dz$$

$$\Rightarrow K_0 = \{x^2+y^2+z^2<R^2\} \setminus \{x^2+y^2+(z-d)^2<R_1^2\}$$

$$N = \int_{K} \frac{\varrho_0}{|A-X|}\,dx\,dy\,dz + \int_{K_1} \frac{\varrho_1-\varrho_0}{|A-X|}\,dx\,dy\,dz$$

kde $K = \{x^2+y^2+z^2<R^2\}$

Tedy máme počítat integrál:

$$\int_{-R}^{R} \left(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \left(\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} \frac{\rho_0}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}\, dz \right) dy \right) dx \; +$$

$$\int_{-R_1}^{R_1} \left(\int_{-\sqrt{R_1^2-x^2}}^{\sqrt{R_1^2-x^2}} \left(\int_{b-\sqrt{R_1^2-x^2-y^2}}^{b+\sqrt{R_1^2-x^2-y^2}} \frac{\rho_1-\rho_0}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}\, dz \right) dy \right) dx$$

Hezké spočítat!

Kap III: 55/39

Máme spočítat $\displaystyle\int_{\mathbb{R}^3} \frac{\rho}{|X-A|}\, dx\, dy\, dz = \int_{\mathbb{R}^3} \frac{\rho}{\sqrt{x^2+y^2+z^2}}\, dx\, dy\, dz =$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \frac{\rho}{\sqrt{x^2+y^2+z^2}}\, dx \right) dy \right) dz = \iiint_{-\infty}^{+\infty} \frac{\rho}{\sqrt{y^2+z^2}\sqrt{1+\left(\frac{x}{\sqrt{y^2+z^2}}\right)^2}}\, dx\, dy\, dz$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\rho}{\cosh \alpha} \cdot \cosh\alpha\, d\alpha\, dy\, dz = +\infty.$$

$\frac{x}{\sqrt{y^2+z^2}} = \sinh \alpha$