DÚ 10 2, 3, v oboru konvergence sečtěte $\cancel{\sum_{n=1}^{\infty} n^2 x^{n-1}}$

MAFO34

Sečtěte $\sum_{n=1}^{+\infty} \frac{(-1)^n n^2}{n!}$, spočtěte $\int_0^1 \frac{\lg(1+x)}{x}\, dx$

Besselova rovnice: $t^2 y'' + t y' + (t^2 - n^2) y = 0$. Besselova fce 1. druhu

Pomocí rozkladu integrandu do mocninné řady spočtěte

$$\int_0^1 \cancel{\lg x \lg(1-x)}\, dx$$

$$J_n(t) = a_0 t^n \left(1 + \sum_{i=1}^{\infty} \frac{(-1)^i t^{2i}}{2 \cdots 2i (2n+2) \cdots (2n+2i)}\right)$$

$$a_0 = \frac{1}{2^n n!} \quad = \sum_{s=0}^{+\infty} \frac{(-1)^s (t/2)^{n+2s}}{s!(n+s)!}$$

Ad 2) $\sum_{n=1}^{+\infty} a^{n^2} z^n$; $a \in \mathbb{R}^+$

a) Poloměr konvergence: $\frac{1}{R} = \limsup_{n \to +\infty} \sqrt[n]{a_n} = \lim_{n \to +\infty} |a^n| =$

$$= \begin{cases} 0 & |a| < 1 \\ 1 & |a| = 1 \\ +\infty & |a| > 1 \end{cases} \Rightarrow R = \begin{cases} +\infty & |a| < 1 \\ 1 & |a| = 1 \\ 0 & |a| \geq 1. \end{cases}$$

b) Na hranici konvergence: $(a = 1)$ $\cancel{(\ldots)}$

$\sum_{n=1}^{+\infty} z^n = \sum_{n=1}^{+\infty} e^{int}$ nekonverguje

$(a = -1)$: $\sum_{n=1}^{+\infty} (-1)^n \cdot e^{int}$ nekonverguje

není splněna nutná podm.

Ad 3) $\sum_{n=1}^{+\infty} \frac{a^n + b^n}{n} z^n$; Bůno: $|a| \geq |b|$

a) poloměr konv.: $\frac{1}{R} = \limsup_{n \to +\infty} \sqrt[n]{\left|\frac{a^n + b^n}{n}\right|} = \lim_{n \to +\infty} |a| \sqrt[n]{\frac{\left|1 + \left(\frac{b}{|a|}\right)^n\right|}{n}}$

$$= |a| \Rightarrow R = \frac{1}{|a|}$$

b) Na hranici konvergence: $z = \frac{1}{|a|} \cdot e^{it}$, $t \in [0, 2\pi]$

$$\sum_{n=1}^{+\infty} \frac{1}{n} \left(\left(\frac{a}{|a|}\right)^n + \left(\frac{b}{|a|}\right)^n \right) \cdot e^{int}$$

Provedu pouze pro $|a| > |b|$: $\sum_{n=1}^{+\infty} \frac{1}{n} \left(\frac{b}{|a|}\right)^n \cdot e^{int}$ konverguje absolutně $\forall t$

$a < 0$: $\sum_{n=1}^{+\infty} \frac{1}{n} (-1)^n \cdot e^{int} = \sum_{n=1}^{+\infty} \frac{1}{n} \cdot e^{in(t-\pi)}$ konverguje dle Dirichlet

$a>0$: $\sum_{n=1}^{+\infty} \frac{1}{n} \cdot e^{int}$ konverguje podle Dirichlet. kr.

pro $t \neq 2k\pi$, $k \in \cancel{\mathbb{N}}\, \mathbb{Z}$

$\sum_{n=1}^{+\infty} n^2 x^{n-1}$ konverguje lok. st. v $U_1(0)$ $\Rightarrow$ lze derivovat

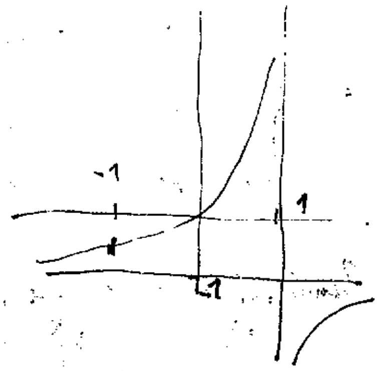
označím

~~$F(x)$~~

~~$(F(x) \cdot x)' = \sum_{n=1}^{+\infty}$~~

Označ: $\sum_{n=1}^{+\infty} x^n = F(x)$. Pak konverguje v $U_1(0)$.

v $U_1(0)$: $\quad F'(x) = \sum_{n=1}^{+\infty} n x^{n-1}$



$$x \cdot F'(x) = \sum_{n=1}^{+\infty} n x^n$$

$$(x \cdot F'(x))' = \sum_{n=1}^{+\infty} n^2 x^{n-1}$$

$$\left(x \cdot \left(\frac{x}{1-x}\right)'\right)' = \left(x \frac{1-x+x}{(1-x)^2}\right)' = \left(\frac{x}{(1-x)^2}\right)' =$$

$$= \frac{(1-x)^2 + 2(1-x)\cdot x}{(1-x)^4} = \frac{1-x+2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}$$

Tedy: $\sum_{n=1}^{+\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3}$ pro $x \in (-1,1)$

Dokážeme něco pro $x = -1$?

$$\sum_{n=1}^{+\infty} \frac{(-1)^n n^2}{n!} = (\tilde{R})$$

Víme: $e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$

Tedy $\sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n = e^{-x}$ a poloměr konvergence je $+\infty$

$$\Rightarrow (e^{-x})' = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n!} x^{n-1} \cdot n$$

Proč!!!

$$\left((e^{-x})' x\right)' = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n!} \cdot n^2 \cdot x^{n-1}$$

$$\parallel$$

$$(-x e^{-x})' = -1 e^{-x} + (+x) \cdot e^{-x} = e^{-x}(x-1) \quad \forall x \in \mathbb{R}$$

Tedy: $\sum_{n=1}^{+\infty} \frac{(-1)^n n^2}{n!} = 0$

$$\int_0^1 \frac{\lg(1+x)}{x}\,dx = \int_0^1 \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} \cdot (-1)^{n+1}\,dx = \sum_{n=1}^{+\infty} \frac{1}{n} \int_0^1 x^{n-1}\,dx\,(-1)^{n+1} =$$

Řada konverguje stejnoměrně

$$= \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{n} \cdot \frac{1}{n} = \cancel{\sum_{n=1}^{+\infty} (-1)^{n+1} \left(\frac{1}{n-1} - \frac{1}{n}\right)}$$

$$= \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{n^2}$$

Sečteme: $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{n^2} x^n = F(x)$

$$\left(F'(x) \cdot x\right)' = \sum_{n=1}^{+\infty} (-1)^{n+1} x^{n-1} = \frac{1}{1+x}$$

$F'(x) \cdot x = \lg(1+x) \Rightarrow F'(x) = \frac{\lg(1+x)}{x}$ ale neumím najít p. f.