DESCRIPTIVE HIERARCHY OF COMPLEX BANACH SPACES

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ABSTRACT. We study the descriptive hierarchy of complex Banach spaces and the associated classes of compact spaces. We show that up to weakly Lindelöf determined spaces and Corson compacta the theory is completely paralel to that of real Banach spaces. For Valdivia compact spaces and related classes of Banach spaces the situation is more difficult. We prove analogues of some results known for real spaces and formulate several open problems.

1. INTRODUCTION

By the descriptive hierarchy we mean the following classes of Banach spaces ordered by inclusion:

separable spaces
$\bigcap$
weakly compactly generated spaces
$\bigcap$
subspaces of weakly compactly generated spaces
$\bigcap$
weakly K-analytic spaces
$\bigcap$
weakly countably determined (Vašák) spaces
$\bigcap$
weakly Lindelöf determined spaces
$\bigcap$
1-Plichko spaces
$\bigcap$
Plichko spaces

A natural complement of these classes are the following classes of compact spaces, again ordered by inclusion.

metrizable compacta
$\bigcap$
Eberlein compacta
$\bigcap$
Talagrand compacta
$\bigcap$
Gul'ko compacta
$\bigcap$
Corson compacta
$\bigcap$
Valdivia compacta

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The definitions of these classes will be recalled in the next two sections. The above classes of Banach spaces can be defined both for real and complex Banach spaces, however many results in the literature explicitely or implicitely deal only with the real spaces. One of the reasons is the fact that any weakly countably determined Banach space X is weak Asplund (i.e., any real-valued convex continuous function on X is Gâteaux differentiable at points of a dense G_δ subset of X, see e.g. [7]) and differentiability is studied on real Banach spaces. However, it is natural to ask whether the study of the respective hierarchy for complex spaces gives something new. It is the aim of this paper to address this question.

It turns out that up to weakly Lindelöf determined spaces and Corson compacta nothing new happen, the complex hierarchy is completely analogous to the real one. It is witnessed by Section 2. However, for Valdivia compacta and 1-Plichko spaces the situation is not so easy as witnessed by Section 3.

By a Banach space we mean either a real or a complex Banach space, unless one of these possibilities is explicitely chosen. If X is a Banach space and $A \subset X$, then $\operatorname{span} A$ denotes the set of all linear combinations (complex ones if X is complex, real ones if X is real) of elements of A. If X is complex, we denote by $\operatorname{span}_C A$ the set of all complex linear combinations of elements of A and by $\operatorname{span}_R A$ the set of all real linear combinations of elements of A. Obviously $\operatorname{span}_C A = \operatorname{span}_R(A \cup iA)$.

If X is a complex Banach space, we denote by X_R the space X considered as a real space. The basic tool for most results are the following two easy propositions. The first one compares the weak topologies of X and X_R and the weak* topologies on the respective duals. The second one deals with spaces of continuous functions. Note that for any topological space K we denote by $C(K,\mathbb{R})$ the space of real-valued continuous functions on K and by $C(K,\mathbb{C})$ the space of all complex-valued convex functions on K. If K is compact, we consider on these spaces the supremum norm making $C(K,\mathbb{R})$ a real Banach space and $C(K,\mathbb{C})$ a complex Banach space. By τ_p we denote the topology of pointwise convergence.

Proposition 1.1. *Let X be a complex Banach space.*

- *The identity X onto X_R is a real-linear, isometric and weak-to-weak homeomorphic map.*
- *The mapping $\phi : X^* \to X_R^*$ defined by $\phi(\xi) = \operatorname{Re} \xi$, $\xi \in X^*$, is a real-linear, isometric and weak*-to-weak* homeomorphic map.*

Proof. It is clear from the definition of X_R that the identity from X to X_R is a real-linear isometry.

Let $\xi \in X^*$. Set $\eta = \operatorname{Re} \xi$. Then clearly $\eta \in X_R^*$ and $\|\eta\| \le \|\xi\|$. Moreover, it is easy to check that we have $\xi(x) = \eta(x) - i\eta(ix)$ for each $x \in X$. Let $x \in X$ be such that $\|x\| \le 1$. Choose $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $\alpha\xi(x) = |\xi(x)|$. Then $|\xi(x)| = \alpha\xi(x) = \xi(\alpha x) = \eta(\alpha x) - i\eta(\alpha i x) = \eta(\alpha x) \le \|\eta\|$. Therefore $\|\eta\| = \|\xi\|$. So the mapping ϕ is a real-linear isometry.

Let us further show that the identity of X onto X_R is a weak-to-weak homeomorphism. Let $\eta \in X_R^*$. Then $\eta = \operatorname{Re} \xi$ for some $\xi \in X^*$. As ξ is continuous on (X, w), $\eta = \operatorname{Re} \xi$ is continuous on (X, w) as well. Conversely, let $\xi \in X^*$. Then both $\operatorname{Re} \xi$ and $\operatorname{Im} \xi$ belong to X_R^* and hence are continuous on (X_R, w). Therefore $\xi = \operatorname{Re} \xi + i \operatorname{Im} \xi$ is continuous on (X_R, w), too.

Finally let us show that ϕ is a weak*-to-weak* homeomorphism. The continuity is obvious. Further, let us consider a net ξ_τ of elements of X^* and $\xi \in X^*$ such that $\phi(\xi_\tau)$ weak* converges to $\phi(x)$. It means that $\phi(\xi_\tau)(x)$ converges to $\phi(\xi)(x)$ for each $x \in X$. It follows that $\phi(\xi_\tau)(ix)$ converges to $\phi(\xi)(ix)$ for each $x \in X$ as well, and hence $\xi_\tau(x) = \phi(\xi_\tau)(x) - i\phi(\xi_\tau)(ix)$ converges to $\xi(x) = \phi(\xi)(x) - i\phi(\xi)(ix)$. Therefore ξ_τ weak* converges to ξ.

This completes the proof. ∎

Proposition 1.2. *Let K be a compact space.*

- *Spaces $C(K,\mathbb{C})_R$, $C(K,\mathbb{R})^2$ and $C(K\times\{0,1\},\mathbb{R})$ are isomorphic (as real Banach spaces).*
- *Spaces $(C(K,\mathbb{C}),\tau_p)$, $(C(K,\mathbb{R}),\tau_p)^2$ and $(C(K\times\{0,1\},\mathbb{R}),\tau_p)$ are real-linearly homeomorphic.*

Proof. Define the mappings $T_1 : C(K,\mathbb{C}) \to C(K,\mathbb{R})^2$ and $T_2 : C(K,\mathbb{R})^2 \to C(K\times\{0,1\},\mathbb{R})$ by

$$\begin{aligned} T_1(f) &= (\operatorname{Re} f, \operatorname{Im} f), && f \in C(K,\mathbb{C}),\\ T_2(f_0,f_1)(x,i) &= f_i(x), && f_0,f_1 \in C(K,\mathbb{R}), x\in K, i\in\{0,1\}. \end{aligned}$$

These mappings are clearly real-linear isomorphisms (of real vector spaces). It is obvious that T_1 is a $\tau_p \to \tau_p\times\tau_p$-homeomorphism and T_2 is a $\tau_p\times\tau_p\to\tau_p$-homeomorphism.

Further, for each $f\in C(K,\mathbb{C})$ we have

$$\max\{\|\operatorname{Re} f\|,\|\operatorname{Im} f\|\} \le \|f\| \le \sqrt{\|\operatorname{Re} f\|^2+\|\operatorname{Im} f\|^2}.$$

Finally, if $(f_0,f_1)\in C(K,\mathbb{R})^2$, then $\|T_2(f_0,f_1)\| = \max\{\|f_0\|,\|f_1\|\}$. Therefore T_1 are T_2 are isomorphisms of real Banach spaces. □

2. On the descriptive hierarchy up to weakly Lindelöf determined spaces and Corson compacta

In this section we collect results showing that the descriptive hierarchy of complex Banach spaces up to weakly Lindelöf determined spaces and Corson compacta is completely paralel to that of real spaces. All the results of this section are easy and are proved essentially by quoting the results on real spaces and Propositions 1.1 and 1.2. However, we give these simple proofs for the sake of completeness.

We start by the following obvious results on separable spaces and metrizable compacta.

Theorem 2.1. *Let X be a complex Banach space. The following assertions are equivalent.*

1. *X is separable.*
2. *X_R is separable.*
3. *(B_{X^*},w^*) is metrizable.*
4. *$(B_{X_R^*},w^*)$ is metrizable.*

Proof. The equivalence 1$\Longleftrightarrow$2 is obvious, 3$\Longleftrightarrow$4 follows from Proposition 1.1 and 2$\Longleftrightarrow$4 is a classical fact [8, Proposition 62 and Exercise 3.48]. □

Theorem 2.2. *Let K be a compact space. The following assertions are equivalent.*

1. *K is metrizable.*
2. *$C(K,\mathbb{R})$ is separable.*
3. *$C(K,\mathbb{C})$ is separable.*

Proof. The equivalence 1$\Longleftrightarrow$2 is a well-known fact, see [8, Exercise 3.47], 2$\Longleftrightarrow$3 follows from Proposition 1.2. □

We continue by the next class. A Banach space X is called *weakly compactly generated* (or, shortly *WCG*) if there is a weakly compact $K\subset X$ with $\operatorname{span} K$ dense in X. The dual class of compact spaces is that of *Eberlein compacta.* Recall that a compact K is Eberlein if it is homeomorphic to a subset of a Banach space equipped with the weak topology. Notice that it does not matter whether the Banach space in the question is taken real or complex, as witnessed by the following proposition.

Proposition 2.3. *Let K be a compact space. The following are equivalent.*

(1) *There is a real Banach space X such that K is homeomorphic to a subset of (X, w).*
(2) *There is a complex Banach space X such that K is homeomorphic to a subset of (X, w).*

Proof. If X is a complex Banach space and K is homeomorphic to a subset of (X, w), then K is homeomorphic to a subset of (X_R, w) by Proposition 1.1. Hence $2 \implies 1$ is proved.

To prove the inverse implication suppose that X is a real Banach space and K is homeomorphic to a subset of (X, w). Then X is canonically isometric to a subspace of $C((B_{X^*}, w^*), \mathbb{R})$ and hence also to a subspace of $C((B_{X^*}, w^*), \mathbb{C})_R$. Hence K is homeomorphic to a subset of $(C((B_{X^*}, w^*), \mathbb{C})_R, w)$ and, by Proposition 1.1, also to a subset of $(C((B_{X^*}, w^*), \mathbb{C}), w)$. □

The following two theorems show the relationship of complex and real WCG spaces.

Theorem 2.4. *Let X be a complex Banach space. The following assertions are equivalent.*
(1) *X is WCG.*
(2) *X_R is WCG.*
(3) *There is $M \subset X$ such that $\overline{\operatorname{span}_C M} = X$ and $(\xi(x))_{x\in M} \in c_0(M)$ for each $\xi \in X^*$.*
(4) *There is $M \subset X$ such that $\overline{\operatorname{span}_R M} = X$ and $(\eta(x))_{x\in M} \in c_0(M)$ for each $\eta \in X_R^*$.*

Proof. The implication $2 \implies 4$ follows from [7, Theorem 1.2.5].

$4 \implies 2$ Any one-to-one sequence in M obviously weakly converges to 0. It is then easy to see that $M \cup \{0\}$ is weakly compact. Thus X_R is WCG.

$1 \implies 2$ Let X be WCG. There is $K \subset X$ weakly compact with $\overline{\operatorname{span}_C K} = X$. By Proposition 1.1 the set $K \cup iK$ is weakly compact in X_R. As $\operatorname{span}_R(K \cup iK) = \operatorname{span}_C(K)$, the space X_R is WCG.

$2 \implies 1$ Let X_R be WCG. There is $K \subset X_R$ weakly compact with $\overline{\operatorname{span}_R K} = X$. By Proposition 1.1 K is weakly compact in X. Clearly $\operatorname{span}_C K \supset \operatorname{span}_R K$, hence X is WCG.

$3 \implies 4$ Let M be such a set. Put $M' = M \cup iM$. Then $\overline{\operatorname{span}_R M'} = X$.

Obviously $(\operatorname{Re}\xi(x))_{x\in M} \in c_0(M)$ for each $\xi \in X^*$.

Further, $(\operatorname{Re}\xi(x))_{x\in iM} = (\operatorname{Re}(i\xi(-ix)))_{x\in iM} = -(\operatorname{Im}\xi(-ix))_{x\in iM} \in c_0(iM)$ for each $\xi \in X^*$.

Thus $(\operatorname{Re}\xi(x))_{x\in M'} \in c_0(M')$ for each $\xi \in X^*$, therefore by Proposition 1.1 $(\eta(x))_{x\in M'} \in c_0(M')$ for each $\eta \in X_R^*$.

$4 \implies 3$ Clearly $\overline{\operatorname{span}_C M} = X$. Further, if $\xi \in X^*$, then

$$(\xi(x))_{x\in M} = (\operatorname{Re}\xi(x))_{x\in M} + i(\operatorname{Im}\xi(x))_{x\in M} \in c_0(M)$$

as both $\operatorname{Re}\xi$ and $\operatorname{Im}\xi$ belong to X_R^*. □

WCG spaces are stable to taking complemented subspaces but not to taking general closed subspaces [14]. Therefore the class of Banach spaces isomorphic to a subspace of a WCG space are considered. The relationship of the real and complex case is contained in the following theorem.

Theorem 2.5. *Let X be a complex Banach space. The following assertions are equivalent.*
(1) *X is isomorphic to a subspace of a (complex) WCG space.*
(2) *X_R is isomorphic to a subspace of a (real) WCG space.*
(3) *(B_{X^*}, w^*) is an Eberlein compactum.*
(4) *$(B_{X_R^*}, w^*)$ is an Eberlein compactum.*

Proof. The equivalence 2$\iff$4 is a known fact which can be proved by the standard procedure: If $(B_{X_R^*}, w^*)$ is Eberlein, then $C((B_{X_R^*}, w^*), R)$ is WCG by Theorem 2.7 below and X_R is canonically isometric to a subspace of that space. Conversely, let Y be a real WCG Banach space and Z be a subspace of Y isomorphic to X_R. Then (B_{Y^*}, w^*) is Eberlein by [7, Theorem 1.2.3]. Then (B_{Z^*}, w^*) is a continuous image of (B_{Y^*}, w^*) and hence it is Eberlein by [6]. As Eberlein compacta are stable to taking closed subsets, the assertion 4 immediately follows.

The equivalence 3$\iff$4 follows from Proposition 1.1.

$1 \implies 2$ Let Y be a complex WCG space, Z a subspace of Y isomorphic to X. Then Y_R is a real WCG space by Theorem 2.4, Z_R is a subspace of Y_R isomorphic to X_R.

$3 \implies 1$ X is canonically isometric to a subspace of $C((B_{X^*}, w^*), \mathbb{C})$. The latter space is WCG by Theorem 2.7 below. □

As said above, there is a subspace of a WCG space which itself is not WCG. The first example is due to Rosenthal [14], another one to Argyros [3] (see [7, Section 1.6]). It is usual to consider them as real spaces but the same examples can serve as complex examples.

Example 2.6. *There are non-WCG complex Banach spaces isomorphic to a subspace of a (complex) WCG space.*

Proof. By [14] there is a finite measure μ and a subspace $X \subset L_1(\mu)$ which is not WCG. (Notice that $L_1(\mu)$ is WCG.) If these spaces are considered real (i.e., $L_1(\mu)$ consists of real functions), then $X + iX$ is a complex non-WCG subspace of the complex WCG space $L_1(\mu)$ (consisting of complex functions). Indeed, if $X + iX$ was WCG, then $(X + iX)_R$ would be WCG as well (Theorem 2.4). However, $(X + iX)_R$ is isomorphic to $X \times X$ and X is a complemented subspace of $X \times X$. Hence X would be WCG, a contradiction.

The same could be done for the example of Argyros - it is constructed as a subspace of a space $C(K, \mathbb{R})$ for an Eberlein compact K. We can consider $C(K, \mathbb{C})$ and proceed in the same way as above. □

The case of spaces of continuous functions is described in the following theorem.

Theorem 2.7. *Let K be a compact space. The following assertions are equivalent.*

(1) *K is Eberlein.*
(2) *$C(K, \mathbb{R})$ is WCG.*
(3) *$C(K, \mathbb{C})$ is WCG.*
(4) *$C(K, \mathbb{R})$ is isomorphic to a subspace of a WCG space.*
(5) *$C(K, \mathbb{C})$ is isomorphic to a subspace of a WCG space.*

Proof. The equivalence 1$\iff$2 is a well-known fact, see [7, Theorem 1.2.4].

Let us show 2$\iff$3. By Theorem 2.4 $C(K, \mathbb{C})$ is WCG if and only if $C(K, \mathbb{C})_R$ is WCG. By Proposition 1.2 $C(K, \mathbb{C})_R$ is isomorphic to $C(K, \mathbb{R})^2$. Hence the result easily follows (as WCG space are stable to finite products and complemented subspaces).

The implication $3 \implies 5$ is trivial. To see $5 \implies 4$ use Theorem 2.5 and the observation that $C(K, \mathbb{R})$ is isometric to a subspace of $C(K, \mathbb{C})_R$.

Finally, the implication $4 \implies 1$ is a well-known fact: If $C(K, \mathbb{R})$ is a subspace of a WCG space, then the dual unit ball is Eberlein by Theorem 2.5 and K is homeomorphic to a subset of the dual unit ball and hence is itself Eberlein. □

Note that although in the proof of Theorem 2.5 we refer to Theorem 2.7 and vice versa, the proofs are correct. Indeed, the equivalences $1 \iff 2 \iff 3$ of Theorem 2.7 are independent on Theorem 2.5. In the proof of Theorem 2.5 we use just these equivalences. The remaining equivalences of Theorem 2.7 follow from Theorem 2.5.

The next classes are those of weakly K-analytic Banach spaces and weakly countably determined Banach spaces and the associated classes of Talagrand and Gul'ko compacta. Recall that a Banach space X is *weakly K-analytic* if (X, w) is K-analytic (see [7, Section 4.1]); and X is *weakly countably determined* (or, shortly *WCD*) if (X, w) is K-countably determined (see [7, Section 7.1]). Further, a compact K is *Talagrand*, if $(C(K, \mathbb{R}), \tau_p)$ is K-analytic; and K is *Gul'ko* if $(C(K, \mathbb{R}), \tau_p)$ is K-countably determined. We have the following four theorems. The proofs of the first two of them are completely analogous to the proofs of the last two.

Theorem 2.8. *Let X be a complex Banach space. The following assertions are equivalent.*

(1) *X is weakly K-analytic.*
(2) *X_R is weakly K-analytic.*
(3) *(B_{X^*}, w^*) is a Talagrand compact.*
(4) *$(B_{X_R^*}, w^*)$ is a Talagrand compact.*

Theorem 2.9. *Let K be a compact space. The following assertions are equivalent.*

(1) *K is Talagrand.*
(2) *$C(K, \mathbb{R})$ is weakly K-analytic.*
(3) *$C(K, \mathbb{C})$ is weakly K-analytic.*
(4) *$(C(K, \mathbb{C}), \tau_p)$ is K-analytic.*

Theorem 2.10. *Let X be a complex Banach space. The following assertions are equivalent.*

(1) *X is WCD.*
(2) *X_R is WCD.*
(3) *(B_{X^*}, w^*) is a Gul'ko compact.*
(4) *$(B_{X_R^*}, w^*)$ is a Gul'ko compact.*

Proof. The equivalences $1 \Longleftrightarrow 2$ and $3 \Longleftrightarrow 4$ follow from Proposition 1.1. It is a standard fact that $2 \Longleftrightarrow 4$, see [7, Theorem 7.1.9]. □

Theorem 2.11. *Let K be a compact space. The following assertions are equivalent.*

(1) *K is Gul'ko.*
(2) *$C(K, \mathbb{R})$ is WCD.*
(3) *$C(K, \mathbb{C})$ is WCD.*
(4) *$(C(K, \mathbb{C}), \tau_p)$ is K-countably determined.*

Proof. The equivalence $1 \Longleftrightarrow 2$ is a standard result, see [7, Theorem 7.1.8]. It follows from Proposition 1.2 that $1 \Longleftrightarrow 4$. The equivalence $2 \Longleftrightarrow 3$ follows from Propositon 1.2 and Theorem 2.10. □

Any Banach space isomorphic to a subspace of a WCG space is weakly K-analytic and any weakly K-analytic space is WCD. The latter follows from the definitions (as any K-analytic topological space is K-countably determined), the former is, via the above theorems, a consequence of the classical result of Talagrand [15]. Further results of Talagrand [16, 17] say that there are compact spaces K_1 and K_2 such that K_1 is Talagrand but not Eberlein and K_2 is Gul'ko but not Talagrand. Hence, by the above theorems, $C(K_1, \mathbb{C})$ is a complex weakly K-analytic space which is not isomorphic to a subset of a WCG space and $C(K_2, \mathbb{C})$ is a complex WCD space which is not weakly K-analytic.

The next step is towards Corson complacta and weakly Lindelöf determined spaces. A compact space K is called *Corson* if it is homeomorphic to a subset of

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq 0\} \text{ is countable}\}.$$

A Banach space X is called *weakly Lindelöf determined* (or, shortly *WLD*) if there is $M \subset X$ with $\operatorname{span} M$ dense in X such that for each $\xi \in X^*$ the set $\{x \in M : \xi(x) \neq 0\}$ is countable. The following theorem shows the relationship of complex and real WLD spaces.

It contains, moreover, a topological characterization of WLD spaces, using the notion of primarily Lindelöf space. Let us recall the definition. If Γ is any set, L_Γ denotes the one-point lindelöfication of the discrete space Γ. I.e., $L_\Gamma = \Gamma \cup \{\infty\}$ where points of Γ are isolated and neighborhoods of ∞ are complements of countable subsets of Γ. A topological space is *primarily Lindelöf* if it is a continuous image of a closed subset of $L_\Gamma^{\mathbb{N}}$ for a set Γ.

Note that any primarily Lindelöf space is Lindelöf and the class of primarily Lindelöf spaces is stable to closed subsets, continuous images, countable products and countable unions (see [2, Section IV.3]).

Theorem 2.12. *Let X be a complex Banach space. The following assertions are equivalent.*

(1) *X is WLD.*
(2) *X_R is WLD.*
(3) *(B_{X^*}, w^*) is Corson.*
(4) *$(B_{X_R^*}, w^*)$ is Corson.*
(5) *(X, w) is primarily Lindelöf.*

Proof. The equivalence 3$\iff$4 follows from Proposition 1.1. The proof of 2$\iff$4 can be found in [12, Proposition 4.1]. The equivalence 2$\iff$5 follows from [4, Proposition 1.2] and Proposition 1.1.

1 $\implies$ 2 Let $M \subset X$ be a set witnessing that X is WLD. Put $M' = M \cup iM$. Then $\overline{\operatorname{span}_R M'} = X$.

Obviously $\{x \in M : \operatorname{Re}\xi(x) \neq 0\} \subset \{x \in M : \xi(x) \neq 0\}$ is countable for each $\xi \in X^*$.

Further, for $x \in iM$ and $\xi \in X^*$ we have $\operatorname{Re}\xi(x) = \operatorname{Re}(i\xi(-ix)) = -\operatorname{Im}\xi(-ix)$. Hence $\{x \in iM : \operatorname{Re}\xi(x) \neq 0\} \subset i\{x \in M : \xi(x) \neq 0\}$ is countable for each $\xi \in X^*$.

Thus $\{x \in M' : \operatorname{Re}\xi(x) \neq 0\}$ is countable for each $\xi \in X^*$, hence by Proposition 1.1 $\{x \in M' : \eta(x) \neq 0\}$ is countable for each $\eta \in X_R^*$. Thus X_R is WLD.

2 $\implies$ 1 Suppose that X_R is WLD. Let $M \subset X$ be a subset witnessing it. Clearly $\overline{\operatorname{span}_C M} = X$. Further, if $\xi \in X^*$, then $\{x \in M : \xi(x) \neq 0\} = \{x \in M : \operatorname{Re}\xi(x) \neq 0\} \cup \{x \in M : \operatorname{Im}\xi(x) \neq 0\}$ is countable as both $\operatorname{Re}\xi$ and $\operatorname{Im}\xi$ belong to X_R^*. Thus X is WLD. □

For Corson compact spaces the situation is more complicated than that for the previous classes. We have two theorems.

Theorem 2.13. *Let K be a compact space. The following assertions are equivalent.*

(1) *K is Corson.*
(2) *$(C(K, \mathbb{R}), \tau_p)$ is primarily Lindelöf.*
(3) *$(C(K, \mathbb{C}), \tau_p)$ is primarily Lindelöf.*

Proof. The equivalence 1$\iff$2 is the result of [13] (see [2, Section IV.3]). The equivalence 2$\iff$3 easily follows from Proposition 1.2. □

If K is Corson, the space $C(K, \mathbb{R})$ need not be WLD by [5, Theorem 3.12]. An additional property is needed - so called property (M). A compact space K has *property (M)* if each Radon probability measure on K is supported by a separable set.

Theorem 2.14. *Let K be a compact space. The following assertions are equivalent.*

(1) *K is a Corson compactum with property (M).*
(2) *$C(K, \mathbb{R})$ is WLD.*

(3) $C(K,\mathbb{C})$ *is WLD.*

Proof. The equivalence 1$\Longleftrightarrow$2 is proved in [5, Theorem 3.5], the equivalence 2$\Longleftrightarrow$3 follows from Proposition 1.2 and Theorem 2.12. □

There is a Corson compact space K with property (M) which is not Gul'ko (see, e.g. [1]). Then $C(K,\mathbb{C})$ is a complex WLD space which is not WCD.

3. Valdivia compacta and associated Banach spaces

In this section we would like to show that for the highest steps of the descriptive hierarchy the relationship between real and complex cases are not so obvious as for the smaller classes. We start by definitions.

A compact space K is *Valdivia* if it is, for a set Γ, homeomorphic to a subset K' of $\mathbb{R}^\Gamma$ with $K'\cap\Sigma(\Gamma)$ dense in K'. A subset $A\subset K$ is called a Σ*-subset* of Γ if there is a homeomorphic injection $h:K\to\mathbb{R}^\Gamma$ with $A=h^{-1}(\Sigma(\Gamma))$. Hence a compact space is Valdivia if and only if it admits a dense Σ-subset.

Valdivia compact spaces are a natural generalization of Corson compact spaces. For example, the ordinal interval $[0,\omega_1]$ and the Tychonoff cube $[0,1]^I$ for I uncountable are Valdivia compact spaces which are not Corson. For basic properties of Valdivia compacta and related Banach spaces we refer to [10]. Here we recall only few properties which will be constantly used. They are proved in [10, Chapter 1].

Proposition 3.1. *Let K be a compact space.*

1. *Any Σ-subset of K is countably compact and Fréchet-Urysohn.*
2. *If $A\subset K$ is a dense countably compact set, then $G\cap A$ is dense in G for each G_δ set $G\subset K$.*
3. *Let A,B be two subsets of K which are countably compact and Fréchet-Urysohn. If $A\cap B$ is dense in K, then $A=B$.*
4. *If $A\subset K$ is a dense Σ-subset of K, then K is the Čech-Stone compactification of A.*

We continue by a characterization of Valdivia compacta or, more precisely, of dense Σ-subsets generalizing Theorem 2.13. By $\tau_p(A)$ we denote the topology of pointwise convergence on A.

Theorem 3.2. *Let K be a compact space and $A\subset K$ a dense subset. The following assertions are equivalent.*

1. *A is a Σ-subset of K.*
2. *A is countably compact and $(C(K,\mathbb{R}),\tau_p(A))$ is primarily Lindelöf.*
3. *A is countably compact and $(C(K,\mathbb{C}),\tau_p(A))$ is primarily Lindelöf.*

Proof. The equivalence 1$\Longleftrightarrow$2 is proved in [9, Theorem 2.1] (see also [10, Theorem 2.5]). To show 2$\Longleftrightarrow$3, just note that $(C(K,\mathbb{C}),\tau_p(A))$ is canonically homeomorphic to $(C(K,\mathbb{R}),\tau_p(A))^2$ (use the mapping T_1 defined in the proof of Proposition 1.2). □

Now we are going to define associated classes of Banach spaces Let X be a Banach space. A subspace $S\subset X^*$ is a Σ*-subspace* of X^* if there is $M\subset X$ with $\operatorname{span} M$ dense in X such that

$$S=\{\xi\in X^*:\{x\in M:\xi(x)\neq 0\}\text{ is countable}\}.$$

A Banach space is called *Plichko* (1-*Plichko*) if X^* admits a norming (1-norming, respectively) Σ-subspace. Recall that $S\subset X^*$ is *norming* if

$$|x|=\sup\{|\xi(x)|:\xi\in S\cap B_{X^*}\},\qquad x\in X$$

defines an equivalent norm on X. If this norm is equal to the original one, S is called 1-*norming*. Note that a subspace $S \subset X^*$ is 1-norming if and only if $S \cap B_{X^*}$ is weak* dense in B_{X^*}.

The following theorem proved in [9, Theorem 2.3] (see also [10, Theorem 2.7]) is a counterpart of Theorem 3.2 in real Banach spaces.

Theorem 3.3. *Let X be a real Banach space and $A \subset B_{X^*}$ be a weak* dense subset. The following assertions are equivalent.*

(1) *There is a (1-norming) Σ-subspace of X^* with $A = S \cap B_{X^*}$.*
(2) *A is a convex symmetric Σ-subset of (B_{X^*}, w^*).*
(3) *A is weak* countably compact and $(X, \sigma(X,A))$ is primarily Lindelöf.*

The topology $\sigma(X,A)$ is the weakest topology on X making all functionals from A continuous. For complex Banach spaces the point (2) should be slightly changed.

Theorem 3.4. *Let X be a complex Banach space and $A \subset B_{X^*}$ be a weak* dense subset. The following assertions are equivalent.*

(1) *There is a (1-norming) Σ-subspace of X^* with $A = S \cap B_{X^*}$.*
(2) *A is a convex Σ-subset of (B_{X^*}, w^*) satisfying $\alpha A = A$ for each $\alpha \in \mathbb{C}$, $|\alpha| = 1$.*
(3) *A is weak* countably compact and $(X, \sigma(X,A))$ is primarily Lindelöf.*

Proof. The implication $1 \Longrightarrow 2$ is obvious.

For the proof of $2 \Longrightarrow 3$ we follow the proof of the respective implication of [10, Theorem 2.7]. Consider the canonical embedding $e : X \to C((B_{X^*}, w^*), \mathbb{C})$ defined by $e(x)(\xi) = \xi(x)$. As $(C((B_{X^*}, w^*), \mathbb{C}), \tau_p(A))$ is primarily Lindelöf by Theorem 3.2 and e is $\sigma(X,A) \to \tau_p(A)$ homeomorphism, it is enough to show that $e(X)$ is $\tau_p(A)$-closed in $C((B_{X^*}, w^*), \mathbb{C})$. Let Ξ be in the $\tau_p(A)$-closure of $e(X)$ in $C((B_{X^*}, w^*), \mathbb{C})$. Then clearly:

- $\Xi(0) = 0$;
- $\Xi|A$ is affine;
- $\Xi(\alpha\xi) = \alpha\Xi(\xi)$ for each $\alpha \in \mathbb{C}$, $|\alpha| = 1$ and $\xi \in A$.

As A is weak* dense in B_{X^*}, we get that Ξ is the restriction of a linear functional. Hence $\Xi \in e(X)$ by the Banach-Dieudonné theorem [8, Corollary 224].

Also the proof of $3 \Longrightarrow 1$ follows the proof of [10, Theorem 2.7]. By a result of Gul'ko (see [13, Proposition 1.4] or [2, Proposition IV.3.10]) there is a continuous one-to-one linear map $T_0' : (C((X, \sigma(X,A)), \mathbb{R}), \tau_p) \to \Sigma(\Gamma)$ for a set Γ. Define $T_0 : (C((X, \sigma(X,A)), \mathbb{C}), \tau_p) \to \mathbb{C}^\Gamma$ by $T_0(f) = T_0'(\operatorname{Re} f) + iT_0'(\operatorname{Im} f)$. Then T_0 is a continuous one-to-one linear map with range in

$$\Sigma_C(\Gamma) = \{x \in \mathbb{C}^\Gamma : \{\gamma \in \Gamma : x(\gamma) \neq 0\} \text{ is countable}\}.$$

Clearly $\operatorname{span} A \subset C((X, \sigma(X,A)), \mathbb{C})$ and the weak* topology on $\operatorname{span} A$ coincide with the topology of pointwise convergence on X. Hence $T_0(A)$ is dense in $T_0(\operatorname{span} A \cap B_{X^*})$. However, $T_0(A)$ is a countably compact subset of $\Sigma_C(\Gamma)$ and hence it is closed in this space (see [10, Lemma 1.8]). It follows that $A = \operatorname{span} A \cap B_{X^*}$. It remains to show that $\operatorname{span} A$ is a Σ-subspace of X^*.

By [9, Lemma 2.18] applied to the space X_R we get that (B_{X^*}, w^*) is the Čech-Stone compactification of A. It follows that T_0 can be extended to a linear map $T : X^* \to \mathbb{C}^\Gamma$ such that $T_{|B_{X^*}}$ is weak* continuous. By Proposition 3.1 $T(B_{X^*})$ is the Čech-Stone compactification of $T(A)$ and hence T is one-to-one. By Banach-Dieudonné theorem [8, Corollary 224] T is weak* continuous. Hence for each $\gamma \in \Gamma$ there is $x_\gamma \in X$ with $T(\xi)(\gamma) = \xi(x_\gamma)$ (see [8, Theorem 55]). Set $M = \{x_\gamma : \gamma \in \Gamma\}$. Then $\operatorname{span} M$ is dense in X (as T is one-to-one). Moreover,

$$\operatorname{span} A = \{\xi \in X^* : \{x \in M : \xi(x) \neq 0\} \text{ is countable}\}.$$

Indeed, the inclusion $\subset$ is clear, the inverse one follows from Proposition 3.1. This completes the proof. $\square$

In a similar way we can characterize norming Σ-subspaces.

Theorem 3.5. *Let X be a Banach space and $S \subset X^*$ a norm-closed norming subspace. Then the following assertions are equivalent.*

1. *S is a Σ-subspace of X^*.*
2. *S is a countable union of weak* countably compact sets and $(X, \sigma(X,S))$ is primarily Lindelöf.*

Proof. Up to changing the norm on X by an equivalent one we can suppose that S is 1-norming.

Then $1 \implies 2$ follows from Theorems 3.3 and 3.4.

Let us show $2 \implies 1$. We have $S = \bigcup_{n\in\mathbb{N}} S_n$ with each S_n weak* countably compact. As S is norm closed, by Baire category theorem there is $\xi \in S$ and $r > 0$ such that $\overline{B(\xi, r)} \cap S_n$ is norm-dense in $\overline{B(\xi, r)} \cap S$. As S is a linear subspace, it follows that $S \cap B_{X^*}$ has a norm-dense weak* countably compact subset D. Then $D = S \cap B_{X^*}$. Indeed, if $\xi \in (S \cap B_{X^*}) \setminus D$, there is a sequence of $d_n \in D$ norm-converging to ξ. Then the sequence $\{d_n\}$ has no weak* cluster point in D, a contradiction with weak* countable compactness of D. We conclude by Theorem 3.3 or 3.4 that S is a Σ-subspace of X^*. $\square$

Now we proceed to the relationships of the complex and real cases.

Proposition 3.6. *Let X be a complex Banach space and $S \subset X^*$ be a linear subspace. Let $\phi : X^* \to X^*_R$ be as in Proposition 1.1.*

- *If S is 1-norming, then $\phi(S)$ is a 1-norming subspace of X^*_R.*
- *If S is norming, then $\phi(S)$ is a norming subspace of X^*_R.*
- *If S is a Σ-subspace of X^*, then $\phi(S)$ is a Σ-subspace of X^*_R.*

Proof. The first assertion follows immediately from Proposition 1.1. To see the second one use the first one together with the fact that S is norming if and only if it is 1-norming for an equivalent norm.

Let us show the last assertion. Let $M \subset X$ be such that $\operatorname{span}_C M$ is dense in X and $S = \{\xi \in X^* : \{x \in M : \xi(x) \neq 0\}$ is countable$\}$. Set $M' = M \cup iM$. Then $\operatorname{span}_R M'$ is dense in X. Further, if $\xi \in X^*$ and $x \in X$, then $\xi(x) = \operatorname{Re}\xi(x) - i\operatorname{Re}\xi(ix)$ (see the proof of Proposition 1.1). Hence $\xi(x) = 0$ if and only if $\operatorname{Re}\xi(x) = 0$ and $\operatorname{Re}\xi(ix) = 0$. Thus

$$S = \{\xi \in X^* : \{x \in M' : \operatorname{Re}\xi(x) \neq 0\} \text{ is countable}\},$$

therefore $\phi(S)$ is a Σ-subspace of X^*_R. $\square$

Theorem 3.7. *Let X be a complex Banach space. Consider the following assertions.*

1. *X is 1-Plichko.*
2. *X_R is 1-Plichko.*
3. *(B_{X^*}, w^*) is a Valdivia compactum.*

Then $1 \implies 2 \implies 3$. If B_{X^*} is the weak closed convex hull of its weak* G_δ-points, then $1 \iff 2$.*

Proof. The implication $2 \implies 3$ easily follows from the definitions and Proposition 1.1, $1 \implies 2$ follows from Proposition 3.6.

Finally, suppose that B_{X^*} is the weak* closed convex hull of it weak* G_δ-points. We will show $2 \implies 1$. Let G denote the set of all weak* G_δ-points of B_{X^*}. Let X_R be 1-Plichko. Then (B_{X^*}, w^*) has a dense convex symmetric Σ-subset A (see Theorem 3.3). Let $\alpha \in \mathbb{C}$ be such that $|\alpha| = 1$. As $x \mapsto \alpha x$ is a homeomorphism of

(B_{X^*}, w^*), αA is also a (convex symmetric) Σ-subset. By Proposition 3.1 $A \cap \alpha A$ contains G, hence also $\operatorname{conv} G$, so $A \cap \alpha A$ is weak* dense in B_{X^*}. It follows from Proposition 3.1 that $A = \alpha A$. Thus, by Theorem 3.4, X is 1-Plichko. $\square$

It is natural to ask whether the converse implication are valid. It turns out that the implication $3 \implies 2$ does not hold even if we suppose that (B_{X^*}, w^*) has a dense set of G_δ points - see Example 3.10 at the end of this section. We do not know whether $2 \implies 1$ holds in general. The following example shows that a converse of Proposition 3.6 is false.

Example 3.8. *There is a complex Banach space X and $M \subset X$ with* $\operatorname{span}_R M$ *dense in X such that the Σ-subset S_R of X_R^* defined by M is* 1*-norming while the Σ-subset S of X^* defined by M is not even weak* dense. Moreover, $A = \phi^{-1}(S_R) \cap B_{X^*}$ is a convex symmetric Σ-subset of B_{X^*} such that $\alpha A \neq A$ for some $\alpha \in C$, $|\alpha| = 1$. (ϕ is the map defined in Proposition 1.1.)*

Proof. Let $X = \ell_1(\Gamma)$ for some uncountable Γ. By e_γ, $\gamma \in \Gamma$ denote the canonical unit vectors. Choose $\gamma_0 \in \Gamma$ and set

$$M = \{e_{\gamma_0}, ie_{\gamma_0}\} \cup \{e_\gamma - e_{\gamma_0} : \gamma \in \Gamma \setminus \{\gamma_0\}\} \cup \{i(e_\gamma + e_{\gamma_0}) : \gamma \in \Gamma \setminus \{\gamma_0\}\}.$$

Then $\operatorname{span}_R M$ is clearly dense in X. Further, X^* can be canonically identified with $\ell_\infty(\Gamma)$.

The Σ-subspace of X^* defined by M is

$$S = \{\xi = (\xi_\gamma)_{\gamma\in\Gamma} : \{x \in M : \xi(x) \neq 0\} \text{ is countable}\}.$$

Suppose that $\xi \in S$. Then there is $\gamma \in \Gamma \setminus \{\gamma_0\}$ such that $\xi(e_\gamma - e_{\gamma_0}) = \xi(i(e_\gamma - e_{\gamma_0})) = 0$. But $\xi(e_\gamma - e_{\gamma_0}) = \xi_\gamma - \xi_{\gamma_0}$ and $\xi(i(e_\gamma + e_{\gamma_0})) = i(\xi_\gamma + \xi_{\gamma_0})$. If both these numbers are 0, necessarily $\xi_{\gamma_0} = 0$. Hence

$$S \subset \{\xi \in \ell_\infty(\Gamma) : \xi(e_{\gamma_0}) = 0\}.$$

The set on the right-hand side is a weak* closed hyperplane, so S is not weak* dense.

The Σ-subspace of X_R^* defined by M is

$$S_R = \{\xi = (\alpha_\gamma + i\beta_\gamma)_{\gamma\in\Gamma} : \{x \in M : \operatorname{Re}\xi(x) \neq 0\} \text{ is countable}\}.$$

Let $\xi = (\alpha_\gamma + i\beta_\gamma)_{\gamma\in\Gamma} \in \ell_\infty(\Gamma)$ and $\gamma \in \Gamma \setminus \{\gamma_0\}$. Then $\operatorname{Re}\xi(e_\gamma - e_{\gamma_0}) = \alpha_\gamma - \alpha_{\gamma_0}$ and $\operatorname{Re}\xi(i(e_\gamma + e_{\gamma_0})) = -\beta_\gamma - \beta_{\gamma_0}$. Thus

$$S_R = \{\xi = (\xi_\gamma)_{\gamma\in\Gamma} : \{\gamma \in \Gamma : \xi_\gamma \neq \overline{\xi_{\gamma_0}}\} \text{ is countable}\}.$$

This subspace is clearly 1-norming.

That $A = \phi^{-1}(S_R) \cap B_{X^*}$ satisfies the required property is obvious. $\square$

On the other hand, for $C(K)$ spaces the conditions (1) and (2) of Theorem 3.7 are equivalent. It is contained, together with other facts on $C(K)$ spaces, in the following theorem.

Theorem 3.9. *Let K be a compact space. Consider the following assertions.*

(1) *K is Valdivia.*
(2_C) *$C(K,\mathbb{C})$ is* 1*-Plichko.*
($2'_C$) *$C(K,\mathbb{C})_R$ is* 1*-Plichko.*
(2_R) *$C(K,\mathbb{R})$ is* 1*-Plichko.*
(3) *$P(K)$ has a dense convex Σ-subset.*
(4_C) *$(B_{C(K,\mathbb{C})}, w^*)$ is Valdivia.*
(4_R) *$(B_{C(K,\mathbb{R})}, w^*)$ is Valdivia.*
(5) *$P(K)$ is Valdivia.*

Then the following implications hold:

$$\begin{array}{ccccccccccc} 1 & \Longrightarrow & 2_C & \Longleftrightarrow & 2'_C & \Longleftrightarrow & 2_R & \Longleftrightarrow & 3 & \Longrightarrow & 4_C \\ & & & & & & & & \Downarrow & & \Downarrow \\ & & & & & & & & 4_R & \Longrightarrow & 5 \end{array}$$

If K has a dense set of G_δ-points, then all these assertions are equivalent.

Proof. By [10, Theorem 5.2] we have $1 \implies 2_R \iff 3 \implies 4_R \implies 5$. Further, $2_C \implies 2'_C$ by Theorem 3.7 and $2_C \implies 4_C$ is clear.

As $P(K) = \{\xi \in C(K,\mathbb{C})^* : \|\xi\| \le 1 \ \&\ \xi(1) = 1\}$ and this set is weak* G_δ in $B_{C(K,\mathbb{C})^*}$, the implications $2'_C \implies 3$ and $4_C \implies 5$ follow from Proposition 3.1.

It remains to show $2_R \implies 2_C$. Suppose $C(K,\mathbb{R})$ is 1-Plichko. Let $M \subset C(K,\mathbb{R})$ be such that $\operatorname{span} M$ is dense and the Σ-subset S defined by M is 1-norming. As $A = S \cap B_{C(K,\mathbb{R})^*}$ is countably compact, for each $f \in C(K,\mathbb{R})$ there is some $\mu \in A$ with $|\mu(f)| = \|f\|$.

$C(K,\mathbb{R})$ can be canonically viewed as a subset of $C(K,\mathbb{C})$, hence alse M is a subset of $C(K,\mathbb{C})$. It is clear that $\operatorname{span}_C M$ is dense in $C(K,\mathbb{C})$, hence M defines a Σ-subset of $C(K,\mathbb{C})^*$. The dual $C(K,\mathbb{C})^*$ can be, due to Riesz theorem, identified with the space of all complex Radon measures on K equipped with the total variation norm. $C(K,\mathbb{R})^*$ are real-valued Radon measures on K. Via this identification, we see that the Σ-subspace of $C(K,\mathbb{C})^*$ defined by M is equal to $S + iS$. So it is enough to show that $S + iS$ is a 1-norming subspace of $C(K,\mathbb{C})^*$.

First note that for any nonempty closed G_δ set $H \subset K$ there is a probability measure carried by H which belongs to S. Indeed, let H be such a set. Then there is a continuous function $f : K \to [0,1]$ with $H = f^{-1}(1)$. Then $\|f\| = 1$, hence there is $\mu \in S \cap B_{C(K,\mathbb{R})^*}$ with $\mu(f) = 1$. Then necessarily μ is a probability carried by H.

Now let $f \in C(K,\mathbb{C})$. Choose some $k \in K$ with $|f(k)| = \|f\|$. Put $H = f^{-1}(f(k))$. Then H is a nonempty closed G_δ subset of K. Let μ be a probability carried by H which belongs to S. Then $\|\mu\| = 1$, $\mu \in S + iS$ and $|\mu(f)| = |f(k)| = \|f\|$. Hence $S + iS$ is 1-norming.

If K has a dense set of G_δ-points, then $5 \implies 1$ by [10, Theorem 5.3]. □

We do not know whether all the assertions in the previous theorem are equivalent without the additional assumption.

Example 3.10. *There is a complex Banach space X isomorphic to $C([0,\omega_1],\mathbb{C})$ such that (B_{X^*}, w^*) is Valdivia but X_R is not 1-Plichko.*

Proof. In [11] a real Banach space Y isomorphic to $C([0,\omega_1],\mathbb{R})$ such that (B_{Y^*}, w^*) is Valdivia but Y is not 1-Plichko is constructed. For the new example we use the same method:

Note that the dual $C([0,\omega_1],\mathbb{C})^*$ is, due to Riesz theorem, identified with the space M of all complex Radon measures on $[0,\omega_1]$. Define $f : M \to \mathbb{C}$ by $f(\mu) = \mu(\{0\}) \cdot |\mu(\{0\})|$. Set

$$A = \{\mu \in M : \|\mu\| \le 1 \ \&\ \mu(\{\omega_1\}) = f(\mu)\},$$
$$B = \{\mu \in M : |\mu|([0,\omega_1)) + |f(\mu)| + |\mu(\{\omega_1\}) - f(\mu)| \le 1\}.$$

Then the following hold.

(a) B is convex and $\alpha B = B$ for each $\alpha \in \mathbb{C}$, $|\alpha| = 1$.
(b) B is weak* closed.
(c) If B_M denotes the unit ball of M, there is $\delta > 0$ with $\delta B_M \subset B \subset B_M$.
(d) A is a dense Σ-subset of (B, w^*).
(e) A is not convex.

Suppose that we already know that (a)–(e) hold. It follows from (a)–(c) that there is an equivalent norm $|\cdot|$ on $C([0,\omega_1],\mathbb{C})$ such that the respective dual unit ball is B. Set $X = (C([0,\omega_1],\mathbb{C}), |\cdot|)$. The dual unit ball is Valdivia by (d). Further, B has a dense set of G_δ points (it follows from [7, Theorem 1.1.3] that $(C([0,\omega_1]),\mathbb{C})_R$ is Asplund, then use [7, Theorems 1.1.1 and 5.1.12]) and hence A is the unique dense Σ-subset of B (by Proposition 3.1). Hence, by (e) (B, w^*) has no convex dense Σ-subset and so X_R is not 1-Plichko (by Theorem 3.3).

It remains to show the assertions (a)–(e). Except for convexity of B they are either easy or they can be derived from the results of [11]. As f is clearly weak* continuous, the assertions (b) and (d) can be proved copying the proof of [11, Lemma 4] and the assertion (c) follows from the proof of [11, Lemma 3]. If $\alpha \in \mathbb{C}$, $|\alpha| = 1$, then $f(\alpha\mu) = \alpha f(\mu)$ for each $\mu \in M$ and hence $\alpha B = B$. Finally, A is not convex, as 0 and $\frac{1}{2}\delta_0 + \frac{1}{4}\delta_{\omega_1}$ belong to A but $\frac{1}{4}\delta_0 + \frac{1}{8}\delta_{\omega_1}$ does not. (Note that δ_x is the Dirac measure supported by x.)

To show that B is convex we cannot use directly [11, Lemma 1] as it heavily uses the functions in question are real. In fact, this lemma is false for complex functions. We will show it using some facts on delta-convex mappings. Let Y and Z be real normed spaces and $F : Y \to Z$ a mapping. The mapping F is said to be *delta-convex* [19] if there is a continuous convex function $f : Y \to \mathbb{R}$ such that $f + \zeta \circ F$ is a continuous convex function on Y for every $\zeta \in Z^*$, $\|\zeta\| = 1$. Such a function f is called a *control function* of F.

We will need the following result on superpositions of delta-convex mappings proved in [19, Proposition 4.1].

Lemma 3.11. *Let X, Y, Z be real normed spaces, $F : X \to Y$ be delta-convex with a control function f, $G : Y \to Z$ be delta-convex with a control function g. Suppose further that G and g are Lipschitz on Y with constants L_G and L_g.*

Then the mapping $G \circ F$ is delta-convex on X with a control function $g \circ F + (L_G + L_g)f$.

Using this lemma we can show the following one.

Lemma 3.12. *Let X and Y be real normed spaces and $F : X \to Y$ be a delta-convex function with a control function $f(x) = \|F(x)\|$. Then the function $H : X \times Y \to \mathbb{R}$ defined by $H(x, y) = \|F(x)\| + \|y - F(x)\|$ is convex.*

Proof. First note that the mapping $Q : X \times Y \to Y$ defined by $Q(x, y) = y - F(x)$ is delta-convex with the control function $\tilde{f}(x, y) = \|F(x)\|$.

Further, the map $G : Y \to \mathbb{R}$ defined by $G(y) = \|y\|$ is convex and 1-Lipschitz. Therefore G is a delta-convex mapping with a control function $g = G$. Using Lemma 3.11 we get that the mapping $G \circ Q(x, y) = \|y - F(x)\|$ is delta-convex with a control function $h(x, y) = \|y - F(x)\| + 2\|F(x)\|$. In particular, $2H = h + G \circ Q$ is convex, hence H is convex, too. □

Lemma 3.13. *The function $\Psi : \mathbb{C} \to \mathbb{C}$ defined by $\Psi(z) = z|z|$ is delta-convex with a control function $\psi(z) = |z|^2$ ($\mathbb{C}$ is considered as the two-dimensional real Hilbert space).*

Proof. First we express Ψ and ψ as maps on $\mathbb{R}^2$. Then

$$\Psi(x, y) = \sqrt{x^2 + y^2} \cdot (x, y), \qquad (x, y) \in \mathbb{R}^2,$$
$$\psi(x, y) = x^2 + y^2, \qquad (x, y) \in \mathbb{R}^2.$$

To show that Ψ is a delta-convex mapping with a control function ψ, we will use the definition. We have to show that $\psi + \xi \circ \Psi$ is convex for each $\xi \in (\mathbb{R}^2)^*$ with $\|\xi\| = 1$. Let $\xi \in (\mathbb{R}^2)^*$ be of norm one. Then there are $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ such that $\xi(x, y) = ax + by$ for $(x, y) \in \mathbb{R}^2$. Hence we need to prove that the function

$$(x, y) \mapsto x^2 + y^2 + (ax + by)\sqrt{x^2 + y^2}$$

is convex on $\mathbb{R}^2$ whenever $a^2 + b^2 = 1$. Due to symmetry (i.e., up to a choice of another ortonormal basis) we may suppose $a = 1$ and $b = 0$. Hence, it remains to show that the function

$$g(x, y) = x^2 + y^2 + x\sqrt{x^2 + y^2}$$

is convex on $\mathbb{R}^2$. The function g is C^∞ on $\mathbb{R}^2 \setminus \{(0, 0)\}$ and hence we can compute the Hess matrix for any point (x, y) except for $(0, 0)$. This matrix is equal to

$$\begin{pmatrix} \dfrac{2\,x^3 + 3\,x\,y^2 + 2\,(x^2 + y^2)^{(3/2)}}{(x^2 + y^2)^{(3/2)}} & \dfrac{y^3}{(x^2 + y^2)^{(3/2)}} \\ \dfrac{y^3}{(x^2 + y^2)^{(3/2)}} & \dfrac{x^3 + 2\,(x^2 + y^2)^{(3/2)}}{(x^2 + y^2)^{(3/2)}} \end{pmatrix}$$

The determinant is equal to

$$3\,\frac{2\,x^2 + 2\,x\,\sqrt{x^2 + y^2} + y^2}{x^2 + y^2} = 3\,\frac{(x + \sqrt{x^2 + y^2})^2}{x^2 + y^2}.$$

This expression is nonnegative and for $y \neq 0$ it is strictly positive. Further,

$$\frac{\partial^2 g}{\partial y^2}(x, y) = \frac{x^3 + 2\,(x^2 + y^2)^{(3/2)}}{(x^2 + y^2)^{(3/2)}}$$

is also nonnegative and for $y \neq 0$ strictly positive. Hence the Hess matrix is (by the Sylvester rule) positive definite whenever $y \neq 0$.

If $y = 0$, then

$$\frac{\partial^2 g}{\partial x^2}(x, y) = \frac{2\,x^3 + 3\,x\,y^2 + 2\,(x^2 + y^2)^{(3/2)}}{(x^2 + y^2)^{(3/2)}}$$

is equal to $\dfrac{2x^3 + 2|x|^3}{|x|^3}$, and hence it is nonnegative. Thus, again by the Sylvester rule, the Hess matrix is positive semidefinite.

It follows that the function g is convex on each line noncontaining $(0, 0)$ and on each half-line starting at $(0, 0)$. To complete the proof that g is convex, it remains to show that the function $u(t) = g(tx, ty)$ is convex on $\mathbb{R}$ for each $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. We already know that any such u is convex on $(-\infty, 0]$ and on $[0, +\infty)$. Further, $u'(0) = 0$ and hence u is convex on $\mathbb{R}$. This completes the proof. □

Now we are ready to complete the proof of Example 3.10. It follows from Lemma 3.13 and Lemma 3.12 that the function $(w, z) \mapsto |z|^2 + |w - z|z||$ is convex on $\mathbb{C}^2$. As $\mu \mapsto (\mu(\{0\}), \mu(\{\omega_1\}))$ is a linear map, the map

$$\mu \mapsto \|f(\mu)\| + \|\mu(\{\omega_1\}) - f(\mu)|$$

is convex on M and hence the set B is clearly convex. □

4. Final remarks and open questions

In this section we comment some open questions mentioned above and give some related problems. First one concerns Theorem 3.9 – are all the conditions equivalent? This was asked, in fact, already in [10, Question 5.10]. We can sum up the question to the following one.

Question 4.1. *Let K be a compact space such that $P(K)$, the space of all Radon probability measures on K equipped with the weak* topology, is a Valdivia compactum. Is K Valdivia, too?*

Another question is related to Theorem 3.7.

Question 4.2. *Let X be a complex Banach space such that X_R is* 1*-Plichko. Is X* 1*-Plichko, too?*

Example 3.8 shows that there may exist 1-norming Σ-subspace of X_R^* such that $\phi^{-1}(S)$ is not a Σ-subspace of X^* (ϕ is the mapping from Proposition 1.1). However, the example is $\ell_1(\Gamma)$ for a set Γ and this space is 1-Plichko – the Σ-subspace generated by the standard basis is 1-norming.

In view of Theorems 3.3 and 3.4 the previous question is equivalent to the following one.

Question 4.3. *Let X be a complex Banach space such that (B_{X^*}, w^*) has a dense convex symmetric Σ-subset. Does (B_{X^*}, w^*) admit another dense Σ-subset A which is convex and satisfies $\alpha A = A$ for each $\alpha \in \mathbb{C}$, $|\alpha| = 1$.*

This question inspires some further questions on the algebraic structure of Valdivia compacta. Namely, let K be a Valdivia compactum and G a group of homeomorphisms of K. Is there a dense Σ-subset A of K which is G-invariant (i.e., $g(A) = A$ for each $g \in G$)? This general question has a negative answer: Let $K = \{0,1\}^I$ where I has cardinality continuum. Then K is Valdivia. By [18] there is a minimal homeomorphism h of K (i.e., all orbits of h are dense in K). Then there is no h-invariant nonempty Σ-subset of K. Indeed, if A is a nonempty h-invariant set, A contains a countable subset dense in K. If A was a Σ-subset, it would be countably closed in K and hence equal to K. However, K is not Corson, as it is not Fréchet-Urysohn.

Hence we will ask more modestly:

Question 4.4. *Let K be a Valdivia compact space and G a finite abelian group of homeomorphisms of K. Is there a G-invariant dense Σ-subspace?*

The positive answer to this question would not solve the previous one, as the group of homeomorphisms $x \mapsto \alpha x$, $|\alpha| = 1$ is infinite and, moreover, the previous question deals with convex sets. However, we do not know answer even to this question and it seems that a positive answer could help to better understand the previous case. In fact, we do not know answer even to the following question.

Question 4.5. *Let K be a Valdivia compact space and $h : K \to K$ be a homeomorphism such that $h \circ h = \mathrm{id}_K$. Is there an h-invariant dense Σ-subset?*

In particular, the following question is open.

Question 4.6. *Let X be a Banach space such that (B_{X^*}, w^*) is Valdivia. Is there a symmetric dense Σ-subset of (B_{X^*}, w^*)?*

Note, that if A is a dense Σ-subset of K and h a homeomorphism of K onto K, then $h(A)$ is a dense Σ-subset, too. Hence, if K has a unique dense Σ-subset, it must be h-invariant. It follows that the method used in [10, Example 6.8], [11] and in Theorem 3.10 above to produce convex Valdivia compacta without dense convex Σ-subsets, cannot be used to produce counterexamples to the mentioned questions, as all these examples are convex Valdivia compacta with a unique non-convex dense Σ-subsets.

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References

[1] K. Alster and R. Pol, *A function space $C(K)$ which is weakly Lindelöf but not weakly compactly generated*, Studia Math. **64** (1979), 279–285.

[2] A.V.Archangel'skij, *Topological spaces of functions*, Kluver Acad. Publ. Dordrecht – Boston – London, 1992.

[3] S. Argyros, *Weakly lindelöf determined Banach spaces not containing ℓ_1*, preprint, 2005.

[4] S. Argyros and S. Mercourakis, *On weakly Lindelöf Banach spaces*, Rocky Mountain J. Math. **23** (1993), 395–446.

[5] S.Argyros, S.Mercourakis and S.Negrepontis, *Functional analytic properties of Corson compact spaces*, Studia Math. **89** (1988), 197–229.

[6] Y. Benjamini, M.E. Rudin and M. Wage, *Continuous images of weakly compact subsets of Banach spaces*, Pacific J. Math. **70** (1977), 309–324.

[7] M.Fabian, *Gâteaux differentiability of convex functions and topology: weak Asplund spaces*, Wiley-Interscience, New York, 1997.

[8] P.Habala, P.Hájek and V.Zizler, *Introduction to Banach spaces*, lecture notes, Matfyzpress, Prague, 1996.

[9] O.Kalenda, *A characterization of Valdivia compact spaces*, Collectanea Math. **51** (2000), no. 1, 59–81.

[10] O.Kalenda, *Valdivia compact spaces in topology and Banach space theory*, Extracta Math. **15** (2000), no. 1, 1–85.

[11] O.Kalenda, *A new Banach space with Valdivia dual unit ball*, Israel J. Math. **131** (2005), 139–147.

[12] J.Orihuela, W.Schachermayer and M.Valdivia, *Every Radon-Nikodým Corson compact is Eberlein compact*, Studia Math. **98** (1991), 157–174.

[13] R.Pol, *On pointwise and weak topology on function spaces*, preprint 4/84, Warszaw Univ., 2005.

[14] H.P. Rosenthal, *The heredity problem for weakly compactly generated Banach spaces*, Compositio Math. **28** (1974), 83–111.

[15] M. Talagrand, *Sur une conjecture de H.H. Corson*, Bull. Sci. Math. **99** (1975), no. 4, 211–212.

[16] M. Talagrand, *Espaces de Banach faiblement K-analytic*, Annals of Math. **110** (1979), 403–438.

[17] M. Talagrand, *A new countably determined Banach space*, Israel J. Math. **47** (1984), 75–80.

[18] S. Turek, *Minimal actions on Cantor cubes*, Bull. Polish Acad. Sci. Math. **51** (2003), no. 2, 129–138.

[19] L.Veselý and L.Zajíček, *Delta-convex mappings between Banach spaces and apllications*, Dissertationes Math. **CCLXXXIX** (1989), 1–52.

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