Lemma IV.5 $\xi \in \mathbb{R}$, $R > 0$

f spojitá na $\{z \in \mathbb{C};\ \operatorname{Re} z \le \xi,\ |z| > R\}$

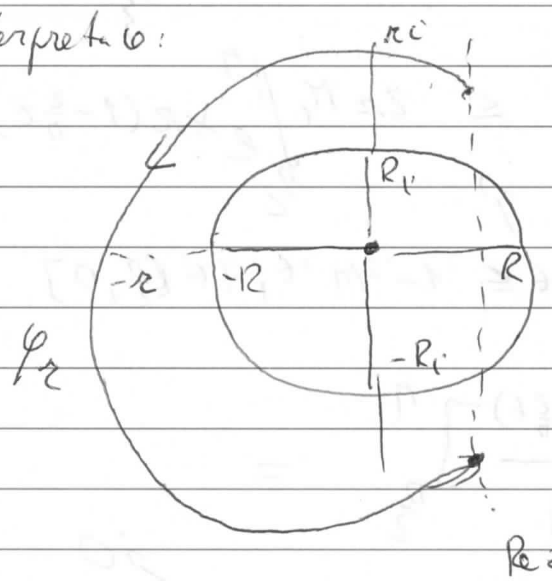
$\lim\limits_{\substack{z\to\infty \\ \operatorname{Re} z \le \xi}} f(z) = 0$. Pro $r > \max(R, |\xi|)$ nechť
$\varphi_r(t) = re^{it},\ t \in [\alpha_r, 2\pi - \alpha_r]$,
kde $\alpha_r \in (0,\pi)$ splňuje $\operatorname{Re}(re^{i\alpha_r}) = \xi$

Pak $\forall x > 0$: $\lim\limits_{r\to\infty} \int_{\varphi_r} f(z)e^{xz}\,dz = 0$.

Pokud navíc $\lim\limits_{\substack{z\to\infty \\ \operatorname{Re} z \le \xi}} z f(z) = 0$, pak to platí i pro $t = 0$, tj.

$\lim\limits_{r\to\infty} \int_{\varphi_r} f = 0$

Dk: Geometrická interpretace:



1) Pro $r > \max\{R, |\xi|\}$ nechť $M_r = \max\{|f(z)|;\ |z| = r,\ \operatorname{Re} z \le \xi\}$

Pak $\lim\limits_{\substack{z\to\infty \\ \operatorname{Re} z \le \xi}} f(z) = 0$ znamená $\lim\limits_{r\to+\infty} M_r = 0$

a $\lim\limits_{\substack{z\to\infty \\ \operatorname{Re} z \le \xi}} z f(z) = 0$ znamená $\lim\limits_{r\to+\infty} r \cdot M_r = 0$.

Nechť $x>0$

Počítejme: $\left|\int\limits_{\varphi_r} f(z)\, e^{xz}\, dz\right| = \left|\int\limits_{\alpha_r}^{2\pi-\alpha_r} f(re^{it})\, e^{xre^{it}} \cdot re^{it}\cdot i\, dt\right|$

VI.2

$$\le \int\limits_{\alpha_r}^{2\pi-\alpha_r} \underbrace{|f(re^{it})|}_{\le M_r} \cdot \underbrace{\left|e^{xre^{it}}\right|}_{= e^{\mathrm{Re}\, xre^{it}}} \cdot \underbrace{|re^{it}\cdot i|}_{=r}\, dt \le$$

$$\le \int\limits_{\alpha_r}^{2\pi-\alpha_r} M_r \cdot e^{xr\cos t} \cdot r\, dt \le r\cdot M_r \int\limits_{\alpha_r}^{2\pi-\alpha_r} e^{xr\cos t}\, dt =$$

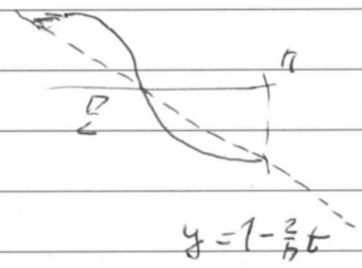
$$= 2rM_r \int\limits_{\alpha_r}^{\pi} e^{xr\cos t}\, dt \xrightarrow{\xi\le 0 \Rightarrow \alpha_r \ge \frac{\pi}{2}} \le 2rM_r \int\limits_{\pi/2}^{\pi} e^{xr\cos t}\, dt$$

$\xi > 0$:

$$= 2rM_r \int\limits_{\alpha_\xi}^{\pi/2} e^{xr\cos t}\, dt + 2rM_r \int\limits_{\pi/2}^{\pi} e^{xr\cos t}\, dt$$

$$2rM_r \int\limits_{\frac{\pi}{2}}^{\pi} e^{xr\cos t}\, dt \le 2rM_r \int\limits_{\frac{\pi}{2}}^{\pi} e^{xr(1-\frac{2}{\pi}t)}\, dt =$$

$\cos t \le 1 - 2/\pi \cdot t, \; t \in [\frac{\pi}{2}, \pi]$



$$= 2rM_r \left[\frac{e^{xr(1-\frac{2}{\pi}t)}}{-xr\,\frac{2}{\pi}}\right]_{\frac{\pi}{2}}^{\pi} =$$

$$= 2r\cdot M_r \cdot \frac{e^{-xr}-1}{-xr\,\frac{2}{\pi}} = 2\overbrace{M_r}^{\to 0} \cdot \overbrace{\frac{1-e^{-xr}}{x}}^{\to \frac{1}{x}} \cdot \frac{\pi}{2} \xrightarrow[r\to\infty]{} 0$$

V případě, že $\xi > 0$, pak

$$2rM_r \int_{\alpha_\xi}^{\frac{\pi}{2}} e^{xr\cos t}\, dt \le 2rM_r \cdot \int_{\alpha_\xi}^{\frac{\pi}{2}} e^{xr\cdot\cos\alpha_\xi}\, dt =$$

$$= 2r\cdot M_r\cdot\left(\tfrac{\pi}{2}-\alpha_\xi\right)\cdot e^{xr\cos\alpha_\xi} = 2rM_r\left(\tfrac{\pi}{2}-\arccos\tfrac{\xi}{r}\right)e^{x\xi}$$

$r\cos\alpha_\xi = \operatorname{Re} re^{i\alpha_\xi} = \xi$

$\cos\alpha_\xi = \frac{\xi}{r}$

$\alpha_\xi = \arccos\frac{\xi}{r}$

$$= \underbrace{2e^{x\xi}\xi\cdot \underset{\downarrow\, 0}{M_r}\cdot \frac{\frac{\pi}{2}-\arccos\frac{\xi}{r}}{\frac{\xi}{r}}}_{\downarrow\, 0 \text{ pro } r\to\infty} \quad \to -\arccos' 0 = 1$$

To dokončuje důkaz, že $\lim_{r\to\infty} \int_{\varphi_r} f(z)e^{xz}\,dz = 0$ pro $x>0$

Případ $x=0$ za předpokladu $\lim_{r\to\infty} rM_r = 0$:

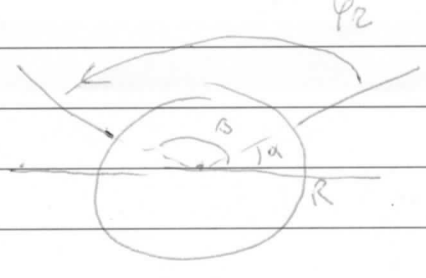
$$\left|\int_{\varphi_r} f\right| \le V(\varphi_r)\cdot M_r \le 2\pi r\cdot M_r \longrightarrow 0 \quad \text{pro } r\to+\infty$$

Lemma IV.6 $\quad 0 \le \alpha < \beta \le \pi$, $R > 0$, f spojitá

na $\{z \in \mathbb{C},\ |z| > R,\ \arg z \in [\alpha, \beta]\}$

$$\lim_{\substack{z\to\infty \\ \arg z \in [\alpha,\beta]}} f(z) = 0$$

Nechť $\varphi_r(t) = re^{it}$, $t \in [\alpha, \beta]$ pro $r > R$



$$\Rightarrow \quad \forall x > 0 \quad \lim_{r\to\infty} \int_{\varphi_r} f(z)\, e^{ixz}\, dz = 0$$

Dk: $\quad M_r := \max\{|f(re^{it})|;\ t \in [\alpha,\beta]\}$, $r > R$

Dle předpokladu $\lim_{r\to+\infty} M_r = 0$

$$\left|\int_{\varphi_r} f(z)\, e^{ixz}\, dz\right| = \left|\int_\alpha^\beta f(re^{it})\, e^{ixre^{it}}\, re^{it}\cdot i\, dt\right| \le$$

$$\underset{\text{V I.2}}{\le} \int_\alpha^\beta \underbrace{|f(re^{it})|}_{\le M_r}\ \left|e^{ixre^{it}}\right|\ \underbrace{|re^{it}\cdot i|}_{=r}\, dt \le r\cdot M_r \cdot \int_\alpha^\beta e^{-xr\sin t}\, dt \le$$

$$\left|e^{ixre^{it}}\right| = e^{\operatorname{Re} ixre^{it}} = e^{-xr\sin t}$$

$$\le r M_r \cdot \int_0^\pi e^{-xr\sin t}\, dt \le 2r M_r \int_0^{\frac{\pi}{2}} e^{-xr\sin t}\, dt \le 2r M_r \int_0^{\frac{\pi}{2}} e^{-xr\frac{2}{\pi}t}\, dt$$

$\sin t \ge \frac{2}{\pi} t$, $t \in [0, \frac{\pi}{2}]$

$$= 2r M_r \left[\frac{e^{-xr\frac{2}{\pi}t}}{-xr\frac{2}{\pi}}\right]_0^{\frac{\pi}{2}} = 2r M_r \frac{e^{-xr} - 1}{-xr\frac{2}{\pi}} =$$

$$= \pi \cdot \underset{\downarrow 0}{M_r} \cdot \underset{\to \frac{1}{x}}{\frac{1 - e^{-xr}}{x}} \xrightarrow{r\to\infty} 0.$$