Lemma 13 $\quad \phi_d(t) = e^{-\frac{1}{2}\|t\|^2}, \; t \in \mathbb{R}^d$. Paž $\phi_d \in \mathcal{S}(\mathbb{R}^d)$

a $\widehat{\phi_d} = \phi_d$

Dk: ① $\phi_d \in \mathcal{S}$, ker velja $\forall \alpha : D^\alpha \phi_d = P \cdot \phi_d$, kjer P je polinom

in $\forall P$ polinom : $\lim_{\|t\| \to \infty} P(x) e^{-\frac{1}{2}\|t\|^2} = 0$

② $d = 1 \quad \Rightarrow \quad \phi_1' = -x \cdot \phi_1$

velja $(\widehat{\phi_1})'(t) = \widehat{-ix\phi_1(t)}(t) = i \cdot \widehat{\phi_1'}(t) = i \cdot i t \widehat{\phi_1}(t)$

$= -t \widehat{\phi_1}(t)$

$\Rightarrow \phi_1$ in $\widehat{\phi_1}$ reši diferencialni enačbi $y' + ty = 0$

ker $\phi_1(0) = 1$

$$\widehat{\phi_1}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{t^2}{2}} dx = 1$$

$$\Rightarrow \widehat{\phi_1} = \phi_1$$

③ $$\widehat{\phi_d}(t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-\|t\|^2/2} \cdot e^{-i\langle t, x\rangle} dx =$$

$$= \prod_{j=1}^{d} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} e^{-x_j^2/2} \cdot e^{-i t_j x_j} dx_j \overset{②}{=} \prod_{j=1}^{d} e^{-t_j^2/2} = e^{-\|t\|^2/2}$$

$$\parallel$$

$$\phi_d(t)$$

V 14 (a) $f \in \mathcal{S} \Rightarrow$ pro $\lambda > 0$

$$\int_{\mathbb{R}^d} f\left(\frac{t}{\lambda}\right) \phi_d(t)\, dm_d(t) = \int_{\mathbb{R}^d} f(y)\, \phi_d(\lambda y) \cdot \lambda^d \, dm_d(y) =$$

$$= \int_{\mathbb{R}^d} f(y) \cdot \lambda^d \hat{\phi}_d(\lambda y)\, dm_d(y) \overset{T8(d)}{=} \int_{\mathbb{R}^d} f(y)\, \widehat{\phi_d\left(\frac{y}{\lambda}\right)}\, dm_d(y) =$$

$$\overset{T8(e)}{=} \int_{\mathbb{R}^d} \hat{f}(y) \cdot \phi_d\left(\frac{y}{\lambda}\right) dm_d(y)$$

$$\lambda \to \infty \Rightarrow \int_{\mathbb{R}^d} f(0)\, \phi_d(t)\, dm_d(t) = \int_{\mathbb{R}^d} \hat{f}(y)\, \phi_d(0)\, dm_d(y)$$

Leb. věta. integr. majoranta $C \cdot \phi_d$ resp. $\hat{f} \cdot$ konst.

$$\Rightarrow f(0) = \int_{\mathbb{R}^d} \hat{f}\, dm_d$$

$$f(x) = (\tau_{-x} f)(0) = \int_{\mathbb{R}^d} \widehat{(\tau_{-x} f)}\, dm_d \overset{T8(a)}{=} \int_{\mathbb{R}^d} \hat{f} \cdot e_x\, dm_d,$$

což jsme chtěli dokázat

(5) $\hat{\hat{f}}(x) = \int_{\mathbb{R}^d} \hat{f}(t) e^{-i\langle t, x\rangle} = \int_{\mathbb{R}^d} \hat{f}(t) e^{i\langle t, -x\rangle}\, dm_d(t) \overset{(a)}{=} f(-x)$ $= \check{f}(x)$

Tedy: $\hat{\hat{f}} = \check{\hat{f}} = f$

Tedy:

$\mathcal{F}: f \mapsto \hat{f}$ je bijekce, inverze $\mathcal{F}^{-1}(f) = \check{f}$

Důsledek 15: Nechť $f \in L^1(\mathbb{R}^d)$ i $\hat{f} \in L^1(\mathbb{R}^d)$

$$\Rightarrow f(x) = \int_{\mathbb{R}^d} \hat{f}(t) e^{i\langle t, x\rangle} dm_d(t) \quad \text{s.v.}$$

Dk: Tavěcupmeno se dá vyjádřit pak:

$$\int_{\mathbb{R}^d} \hat{f}(t) e^{i\langle t,x\rangle} dm_d(t) = \int_{\mathbb{R}^d} \hat{f}(-t) e^{i\langle -t,x\rangle} dm_d(t) =$$
$$= \hat{\check{\hat{f}}}(x)$$

Pro každé $g \in \mathcal{S}(\mathbb{R}^d)$ platí:

$$\int f \cdot \hat{g}\, dm_d = \int \hat{f} \cdot g = \int \hat{f} \cdot \hat{\check{g}} = \int \check{\hat{f}}\, \hat{g} = \int \hat{\check{f}} \cdot \hat{g}$$

T8(e), V14(3), substituce $x \mapsto -x$, T8(e)

$$\Rightarrow \int (f - \hat{\check{f}})\, \hat{g} = 0 \quad \text{pro vš. } g \in \mathcal{S}$$

$f \in L^1$, $\hat{\check{f}} \in C_0 \Rightarrow h := f - \hat{\check{f}} \in L^1_{loc}$. Tvrdím, že $h = 0$ s.v.

Topýno z lemmatu:
$h \in L^1_{loc}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} h \cdot \varphi = 0$ pro každé $\varphi \in \mathbb{R}^d \Rightarrow h = 0$ s.v.

⌈ Sporem: $h \in L^1_{loc}(\mathbb{R}^d)$, h není s.v. nulová. Pak existuje $R > 0$, že h není s.v. nulová na $B(0,R)$. Pak $C := \int_{B(0,R)} |h| \in (0,\infty)$. Zvolme $\delta > 0$ a $M > 0$ tak, aby

$$\int_{\{x \in B(0,R-\delta);\ |h(x)| \le M\}} |h| > \frac{2}{3} C$$

$A :=$ $\{x \in B(0,R-\delta);\ |h(x)| \le M\}$

Definujme funkci $u(x) = \begin{cases} \frac{|h(x)|}{h(x)}, & x \in B(0,R-\delta),\ 0 < h(x) \le M \\ 0 & \text{jinak} \end{cases}$

pro $n \in \mathbb{N}$ nechť $\psi_n := u * \varphi_n$, kde (φ_n) je zhlazovací jádro. Pak $\psi_n \in \mathcal{D}(\mathbb{R}^d)$ spt $\psi_n \subset B(0, R-\delta+\frac{1}{n})$.

Uvažujme $n > \frac{1}{\delta}$. Pak

$$0 = \int_{\mathbb{R}^d} h \cdot \psi_n = \lim_n \int_{\mathbb{R}^d} h \cdot \chi_{B(0,R)} \psi_n = \lim_n \Big(\int_{\mathbb{R}^d} \overbrace{h \chi_A}^{\in L^\infty} \overbrace{\psi_n}^{\to u,\ L^1} + \int_{\mathbb{R}^d} h \chi_{B(0,R)\setminus A} \psi_n \Big) = \int_{\mathbb{R}^d} h \chi_A u + \lim_n \int_{B(0,R)\setminus A} h \overbrace{\psi_n}^{\in[0,1]} \ge \int_A |h| - \int_{B(0,R)\setminus A} |h|$$

$> \frac{\varepsilon}{3}$, spor ⌋