Věta 6:

(i) $\operatorname{spt} \varphi_n \subset B(0,\frac{1}{n})$, $\varphi_n \geq 0$, $\int_{\mathbb{R}^d} \varphi_n = 1$

Γ Připomeňme, že $\varphi_n(x) = n^d \varphi(nx)$, $x \in \mathbb{R}^d$

$\operatorname{spt} \varphi \subset B(0,1) \Rightarrow \operatorname{spt} \varphi_n \subset B(0,\frac{1}{n})$

$\varphi \geq 0 \Rightarrow \varphi_n \geq 0$

$$\int_{\mathbb{R}^d} \varphi_n(x)\,dx = \int_{\mathbb{R}^d} n^d \varphi(nx)\,dx \underset{\substack{x = \frac{y}{n} \\ \text{jakobián} = \frac{1}{n^d}}}{=} \int_{\mathbb{R}^d} n^d \varphi(y) \frac{1}{n^d}\,dy =$$

$$= \int_{\mathbb{R}^d} \varphi(y)\,dy = 1 \quad \lrcorner$$

(ii) $p \in [1,\infty)$, $f \in L^p(\mathbb{R}^d) \Rightarrow f * \varphi_n \to f$ v $L^p(\mathbb{R}^d)$

Buď $f \in L^p(\mathbb{R}^d)$, $\varepsilon > 0$. Dle Lemmatu 5 ex. $\delta > 0$, že pro $\|y\| < \delta$ je $\|\tau_y f - f\|_p < \varepsilon$. Dále ex. j_0, že pro $j \geq j_0$ je $\operatorname{spt} \varphi_j \subset U(0,\delta)$

Zvolme $j \geq j_0$ a $g \in L^q(\mathbb{R}^d)$, $\|g\|_q \leq 1$ $\quad (\frac{1}{q} + \frac{1}{p} = 1)$

Pak $\left| \int_{\mathbb{R}^d} (f * \varphi_j(x) - f(x)) \cdot g(x)\,dx \right| =$

$$= \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(x-y) - f(x)) \varphi_j(y)\,dy \cdot g(x)\,dx \right| \leq$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y) - f(x)|\, \varphi_j(y)\, |g(x)|\,dy\,dx \overset{\text{Fubini}}{=}$$

$$= \int_{\mathbb{R}^d} \varphi_j(y) \cdot \left(\int_{\mathbb{R}^d} |f(x-y) - f(x)|\, |g(x)|\,dx \right) dy$$

$$\leq \int_{\mathbb{R}^d} \varphi_j(y) \cdot \|\tau_y f - f\|_p \|g\|_q\,dy < \varepsilon \int_{\mathbb{R}^d} \varphi_j(y)\,dy = \varepsilon$$

Hölder

(iii) f spojita na $\mathbb{R}^d \Rightarrow \varphi_n * f \overset{loc}{\rightrightarrows} f$ na $\mathbb{R}^d$

f enakomerno spojita na $\mathbb{R}^d \Rightarrow \varphi_n * f \rightrightarrows f$ na $\mathbb{R}^d$

Dk: v dokazu je $h_j = \varphi_j$

f spojita na $\mathbb{R}^d \Rightarrow f * h_j \overset{loc}{\rightrightarrows} f$ na $\mathbb{R}^d$

f enakomerno spojita na $\mathbb{R}^d \Rightarrow f * h_j \rightrightarrows f$ na $\mathbb{R}^d$

Dk: $x \in \mathbb{R}^d$, $j \in \mathbb{N}$:

$$|f * h_j(x) - f(x)| = \left| \int_{\mathbb{R}^d} f(y) h_j(x-y)\,dy - f(x) \int_{\mathbb{R}^d} h_j(x-y)\,dy \right|$$

$$\leq \int_{\mathbb{R}^d} |f(y) - f(x)| \cdot h_j(x-y)\,dy$$

f spojita: $x_0 \in \mathbb{R}^d$, $r > 0$. Ukažimo, da $f * h_j \rightrightarrows f$ na $U(x_0, r)$

$\varepsilon > 0$ poljuben. Ker je f spojita na kompaktni množici $B(x_0, 2r)$, je tam enakomerno spojita, torej obstaja $\delta > 0$, da za vse $x_1, x_2 \in B(x_0, 2r)$, $\|x_1 - x_2\| < \delta$, velja $|f(x_1) - f(x_2)| < \varepsilon$.

Izberimo $j_0 \in \mathbb{N}$, da bo za $j \geq j_0$ bilo supp $h_j \subset U(0, \delta)$

potem za $x \in U(x_0, r)$ je

$$|f * h_j(x) - f(x)| \leq \int_{\mathbb{R}^d} |f(y) - f(x)|\, h_j(x-y)\,dy =$$

$$= \int_{x - U(0,\delta)} |f(y) - f(x)|\, h_j(x-y)\,dy \leq \int_{x - U(0,\delta)} \varepsilon \cdot h_j(x-y) \leq \varepsilon$$

$y \in x - U(0,\delta) \Rightarrow \|y - x\| < \delta \Rightarrow \|y\| < r + \delta < 2r$

torej $|f(y) - f(x)| \leq \varepsilon$

f enakomerno spojita: podobno: $\varepsilon > 0$ ... $\exists\, \delta > 0$, da $\forall x_1, x_2 \in \mathbb{R}^d$,

$\|x_1 - x_2\| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$

$j_0 \in \mathbb{N}$ tak, da za $j \geq j_0$ je supp $h_j \subset U(0, \delta)$

Potem $\forall x \in \mathbb{R}^d$ je $|f * h_j(x) - f(x)| \leq \varepsilon$.