schmidtova reprezentace

$T \in K(H) \setminus \{0\}$

(1) $T^*T$ je samoadjungovaný: $(T^*T)^* = T^*(T^*)^* = T^*T$ (viz V21)

$T^*T$ je kompaktní (V26 (a))

$T^*T \neq 0$ $\left[ \|T^*T\| \overset{T37(v)}{=} \sup_{x \in S_H} |\langle T^*Tx, x\rangle| = \sup_{x \in S_H} |\langle Tx, Tx\rangle| = \|T\|^2 \neq 0 \right]$

Tedy $T^*Tx = \sum_{n=1}^{N} \lambda_n \langle x, x_n\rangle x_n$ dle V38

(2) $W(T^*T) \subset [0,\infty)$ ... $\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 \geq 0$

$\overset{T36}{\Rightarrow} \sigma_p(T) \subset [0,\infty)$, tedy $\lambda_n > 0$ pro každé $n$

(3) Definujme $Sx = \sum_{n=1}^{N} \sqrt{\lambda_n} \langle x, x_n\rangle x_n$, $x \in H$

Pak $S \in K(H)$: $N \in \mathbb{N} \Rightarrow S \in F(H) \subset K(H)$

$N = +\infty \Rightarrow \lambda_n \to 0 \Rightarrow \sqrt{\lambda_n} \to 0$

$S_k x = \sum_{n=1}^{k} \sqrt{\lambda_n} \langle x, x_n\rangle x_n \Rightarrow S_k \in F(H)$

$\|S_k x - Sx\|^2 = \|\sum_{n>k} \sqrt{\lambda_n} \langle x, x_n\rangle x_n\|^2$

$\overset{..}{=} \sum_{n>k} \lambda_n |\langle x, x_n\rangle|^2 \leq (\sup_{n>k} \lambda_n) \sum_{n>k} |\langle x, x_n\rangle|^2$

$\overset{\text{Besselova ner.}}{\leq} (\sup_{n>k} \lambda_n) \|x\|^2$

$\Rightarrow \|S_k - S\|^2 \leq \sup_{n>k} \lambda_n \to 0$

protože $\lambda_n \to 0$

Tedy $S \in K(H)$ $(V26(5))$ 11

$S^* = S$

$$\langle Sx, y\rangle = \langle \sum_{n=1}^{N} \sqrt{\lambda_n} \langle x, x_n\rangle x_n, P_{H_0} y + \sum_{n=1}^{N} \langle y, x_n\rangle x_n \rangle$$

$$= \sum \sqrt{\lambda_n} \langle x, x_n\rangle \overline{\langle y, x_n\rangle} =$$

$$= \langle P_{H_0} x + \sum_{n=1}^{N} \langle x, x_n\rangle x_n, \sum_{n=1}^{N} \sqrt{\lambda_n} \langle y, x_n\rangle x_n \rangle$$

$$= \langle x, Sy\rangle$$

$S^2 = T^* T$

$$S^2 x = S(Sx) = \sum_{n=1}^{N} \sqrt{\lambda_n} \langle Sx, x_n\rangle x_n =$$

$$= \sum_{n=1}^{N} \lambda_n \langle x, x_n\rangle x_n = T^* T x$$

$$\langle Sx, x_n\rangle = \sqrt{\lambda_n} \langle x, x_n\rangle$$

(4) $y_n := \frac{1}{\sqrt{\lambda_n}} T x_n$

$$\|y_n\|^2 = \frac{1}{\sqrt{\lambda_n}^2} \|T x_n\|^2 = \frac{1}{\lambda_n} \langle T x_n, T x_n\rangle =$$

$$= \frac{1}{\lambda_n} \langle T^* T x_n, x_n\rangle = \frac{1}{\lambda_n} \lambda_n \langle x_n, x_n\rangle = 1$$

$m \neq n$: $\langle T x_n, T x_m\rangle = \langle T^* T x_n, x_m\rangle =$

$$= \langle \lambda_n x_n, x_m\rangle = 0$$

Tedy $(y_n)_{n=1}^{N}$ je ON systém

(5) $x \in H \Rightarrow \quad x = P_{H_0}x + \sum_{n=1}^{N} \langle x, x_n \rangle x_n$

$P_{H_0}x \in \ker T^*T$

$\Rightarrow 0 = T^*T(P_{H_0}x)$

tog $\|T P_{H_0}x\|^2 = \langle T P_{H_0}x, T P_{H_0}x \rangle = \langle T^*T P_{H_0}x, P_{H_0}x \rangle$

$= 0$

Tog $Tx = \sum_{n=1}^{N} \langle x, x_n \rangle T x_n = \sum_{n=1}^{N} \sqrt{\lambda_n} \langle x, x_n \rangle y_n$