Comments on the credit test 2 – solutions and evaluation

General comments: I followed the same rules as in the first test – to give a partial credit for a reasonable part of the solution and to require at least one half of the points (i.e., at least 1.5 points). Then three students passed the test and eight students failed. (With the original strict rules, only one student would pass.)

Overall statistics: One student passed both tests, five students passed one test; remaining students have not passed any test yet.

Problem 1

Solution: We need to find all $x \in \mathbb{R}$ such that

$$\arccos \frac{1}{x} < \frac{2}{3}\pi.$$

Since the domain of arccos is the interval $\langle -1, 1\rangle$, arccos is strictly decreasing on this interval and $\frac{2}{3}\pi = \arccos(-\frac{1}{2})$, this inequality is equivalent to

$$\frac{1}{x} \in \langle -1, 1\rangle \qquad \& \qquad \frac{1}{x} > -\frac{1}{2}.$$

Next, $\frac{1}{x} \in \langle -1, 1\rangle$ means that $x \in (-\infty, -1\rangle \cup \langle 1, +\infty)$.

Further, $\frac{1}{x} > -\frac{1}{2}$ holds if and only if either $x > 0$ or $x < -2$.

So, the solution set is $(-\infty, -2) \cup \langle 1, +\infty)$.

Evaluation:

Two students provied a complete almost correct solution - just with a small mistake which I took for misprint. They obtained 0.9 points.

Five students used the condition $\frac{1}{x} \in \langle -1, 1\rangle$. One of them confused then arccos with $\frac{1}{\cos}$ and got 0.2 points. The remaining four have not used the fact that arcsin is decreasing (so, they did not reverse the respective inequality) and instead of the correct value $-\frac{1}{2}$ used another one ($\frac{1}{2}$, $-\frac{\sqrt{3}}{2}$) and made some more mistakes. One of them obtained 0.2 points, the remaining three ones obtained 0.3 points.

Four students obtained 0 points as I have not found in their solutions anything correct. Two of them followed more or less correct way, but made a mistake in almost every step. Third one provided some definition of arccos, some picture and some result (not correct), so I was not able to check the way how this result has been reached. The last one claimed that $\frac{1}{x} < \cos\frac{2}{3}\pi$ (which is not correct, since arccos decreasing, the inequality should be opposite) and then deduced that $\frac{1}{x} < 0.999$, which is a complete nonsense, since $\cos\frac{2}{3}\pi$ equals to $-\frac{1}{2}$, not to 0.999.

Overall comments on this problem: The problem was very similar to the first problem in Test 1. The only difference was that a decreasing function arccos was used instead of the increasing function arcsin and that one should now that $\cos\frac{2}{3}\pi = -\frac{1}{2}$ (which was used instead of $\sin(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$). But some students repeated the same mistakes, even though they could read correct solution and comments on mistakes and evaluation.

Problem 2:

Solution: The solution consists in the following parts - determining the domain of f, the domain of f' and computing the formula for f'.

Domain of f: We have $f(x) = \exp(\operatorname{arctg} x \cdot \log\log(1 + \arcsin x))$. Hence, we have $x \in \langle -1, 1\rangle$ (in order $\arcsin x$ is defined), $1+\arcsin x > 0$ (in order $\log(1+\arcsin x)$ is defined) and $\log(1+\arcsin x) > 0$ (in order $\log\log(1 + \arcsin x)$ is defined). The inequality $\log(1 + \arcsin x) > 0$ is equivalent to $1 + \arcsin x > 1$. If we put it together, we get that $x \in \langle -1, 1\rangle$ and $\arcsin x > 0$, i.e., the domain of f is $(0, 1\rangle$.

The domain of f' is then $(0, 1)$, as no more problems appear.

The formula for f' can be computed by standard rules, so

$$f'(x) = \exp(\operatorname{arctg} x \cdot \log\log(1 + \arcsin x)) \cdot \left(\frac{1}{1+x^2} \cdot \log\log(1 + \arcsin x) \right.$$
$$\left. + \operatorname{arctg} x \cdot \frac{1}{\log(1 + \arcsin x)} \cdot \frac{1}{1 + \arcsin x} \cdot \frac{1}{\sqrt{1-x^2}} \right).$$

Evaluation: Seven students correctly computed the formula for f'. Two of them did not take care of the domain, two students took care but either in a completely wrong way or in a completely insuficcient way. These four students obtained 0.4 points. Two students tried to determine the domain in a correct way, but made quite a lot of mistakes. They obtained 0.5 points. One student correctly determined the domain of f but did not write anything on the domain of f'. This one obtained 0.8 points.

Four students computed the formula for f' in a wrong way. Two of them made a reasonable step to determining the domain, they obtained 0.2 points, the remaining two obtained 0 points.

Overall comments on the problem: I find strange that some students are not able to compute the derivative of a function of the form $u(x)^{v(x)}$, where u and v are some functions, in the standard way. Some students made exactly the same mistakes as in the first test.

Problem 3:

Solution: The solution consists in the following parts - determining the domain of g, drawing a picture of the domain, computing the formulas for partial derivatives and observing that those formulas are valid on the whole domain of g.

The domain of g is described by conditions $xy > 0$ (in order $\log(xy)$ is defined) and $\log(xy) > 0$ (in order to $\log(\log(xy))$ is defined). Since the condition $\log(xy) > 0$ is equivalent to $xy > 1$, the domain of g is equal to $\{[x, y] \in \mathbb{R}^2 : xy > 1\}$.

To make the picture it is enough to observe that it is the set of those pairs $[x, y]$ such that either $x > 0$ and $y > \frac{1}{x}$ or $x < 0$ and $y < \frac{1}{x}$.

Partial derivatives can be computed in the standard way:

$$\frac{\partial g}{\partial x}(x, y) = \frac{1}{\log(xy)} \cdot \frac{1}{xy} \cdot y,$$
$$\frac{\partial g}{\partial y}(x, y) = \frac{1}{\log(xy)} \cdot \frac{1}{xy} \cdot x.$$

It is also clear that these formulas are valid on the whole domain of g. The formulas can be simplified a bit.

Evaluation:

Two students provided a complete correct solution. They obtained 1 point.

One student correctly determined the domain, made the correct picture and correctly computed the derivatives, but only for the case $x > 0$ and $y > 0$. If $x < 0$ and $y < 0$, this approach does not work. This student obtained 0.8 points.

One student correctly determined the domain, made the picture, but the formulas for partial derivatives are wrong. The mistake is essential, but it is possible that it is a misprint. This student obtained 0.6 points.

Five students correctly computed the formulas. Three of them made a reasonable attempt to determine the domain – they obtained 0.5 points. Two of them made some completely wrong comments on the domain – they obtained 0.4 points.

Two students computed the partial derivatives in a wrong way. One of them made a reasonable attempt to determine the domain (and obtained 0.2 points), the other one wrote some strange conditions (some are relevant and some are not relevant) and made a strange picture (and obtained 0 points).

Overall comments on the test:

I understand that anybody can make a mistake. But I assume that when one learns that he or she made a mistake, he or she will read the comments, will redo the computation for himself or herself and will avoid this mistake next time. If somebody repeats the same mistakes in spite of corrections and explanations, I cannot understand this.

In the next test students can expect similar problems which will give them chance to show they are able to avoid identical mistakes.

In particular, let me stress the following things:

• $\arcsin \neq \frac{1}{\sin}$, $\arccos \neq \frac{1}{\cos}$, $\operatorname{arctg} \neq \frac{1}{\operatorname{tg}}$

• $f(x)^{g(x)} = \exp(g(x) \log f(x))$, so both determining the domain and computing the derivative should start from this formula.

• The values of goniometric functions at some canonical points (like 0, π, $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$, $\frac{2}{3}\pi$, $\frac{3}{4}\pi$, $\frac{5}{6}\pi$ etc.) should not be computed numerically on calculators, but should be deduced from several known values and properties of goniometric functions.

• Numerical computation on calculators should be avoided. Results like 0.999 or 1.001 or 0.074 are not interesting for us. It is better to have results like $\frac{1}{2}$, $\frac{\pi}{4}$ or $\sqrt{3}$.