

General comments: No student provided a complete correct solution of two problems. Therefore, if I applied the strict rules, no student would pass the test. I decided to relax the rules – to give a partial credit for reasonable part of the solution and to require at least one half of the points (i.e., at least 1.5 points). Then four students passed the test and eight students failed.

All the students, except for one, provided at least a reasonable part of the solution of one of the problems.

Problem 1

Solution: We need to find all $x \in \mathbb{R}$ such that

$$\arcsin \frac{1}{x} > -\frac{\pi}{4}.$$

Since the domain of \arcsin is the interval $\langle -1, 1 \rangle$, \arcsin is strictly increasing on this interval and $-\frac{\pi}{4} = \arcsin(-\frac{1}{\sqrt{2}})$, this inequality is equivalent to

$$\frac{1}{x} \in \langle -1, 1 \rangle \quad \& \quad \frac{1}{x} > -\frac{1}{\sqrt{2}}.$$

Next, $\frac{1}{x} \in \langle -1, 1 \rangle$ means that $x \in (-\infty, -1) \cup \langle 1, +\infty \rangle$.

Further, $\frac{1}{x} > -\frac{1}{\sqrt{2}}$ holds if and only if either $x > 0$ or $x < -\sqrt{2}$.

So, the solution set is $(-\infty, -\sqrt{2}) \cup \langle 1, +\infty \rangle$.

Evaluation:

Four students arrived to $\frac{1}{x} > -\frac{1}{\sqrt{2}}$. They did not take into account the condition $\frac{1}{x} \in \langle -1, 1 \rangle$ and did not discuss the sign of x . Such students obtained 0.4 points.

One student arrived to $\frac{1}{x} > -\frac{1}{\sqrt{2}}$, correctly discussed the sign of x , but did not take into account the condition $\frac{1}{x} \in \langle -1, 1 \rangle$. This one obtained 0.8 points.

One student followed the complete correct solution, but with some mistakes – instead of $\frac{1}{x} \in \langle -1, 1 \rangle$ was considered only $\frac{1}{x} \leq 1$ and there were mistakes in discussing the sign. This student obtained 0.6 points.

The remaining six students obtained 0 points. Two of them finished at the stage $\arcsin \frac{1}{x} > -\frac{\pi}{4}$. The other four continued after this step but in a completely wrong way (one of them probably confused \arcsin with \arctg , one of them confused \arcsin with $\frac{1}{\sin}$ and two of them wrote a nonsense which I do not understand).

Problem 2:

Solution: This is a completely standard problem on computing a limit. It requires the following steps:

Step 1: Multiply the expression by $\frac{a^2+ab+b^2}{a^2+ab+b^2}$, where $a = \sqrt[3]{n^3 + \cos \frac{2}{n}}$ and $b = \sqrt[3]{n^3 + 1}$. Then we obtain

$$\frac{n^4(\cos \frac{2}{n} - 1)}{a^2 + ab + b^2}$$

Step 2: Compute

$$\lim_{n \rightarrow \infty} n^2 \left(\cos \frac{2}{n} - 1 \right) = \lim_{n \rightarrow \infty} \frac{\cos \frac{2}{n} - 1}{\left(\frac{2}{n} \right)^2} \cdot 4 = -\frac{1}{2} \cdot 4 = -2.$$

(We used Heine theorem and a known limit.)

Step 3: Compute

$$\lim_{n \rightarrow \infty} \frac{n^2}{a^2 + ab + b^2} = \frac{1}{3}$$

by extracting n^2 from the denominator.

Step 4: Conclude that the limit is $-\frac{2}{3}$.

Evaluation: Four students provided a complete correct solution. They obtained 1 point.

Four students followed a correct way of solving the problem to the end, but made some mistakes on the way, which can be viewed as misprints or omissions. They obtained 0.9 points.

One student made steps 1 and 3 but failed to make step 2. This student obtained 0.6 points.

Two students made just step 1. One of them they made some calculations without approaching the result. The other one tried to make step 2 but in a completely wrong way. Both obtained 0.2 points.

One student just made a completely wrong computation. This one obtained 0 points.

Problem 3:

Solution: The solution consists in the following parts - determining the domain of f , the domain of f' and computing the formula for f' .

Domain of f : We have $f(x) = \exp(\sin x \cdot \log \log \frac{1}{x})$. Hence, we have $x \neq 0$ (in order $\frac{1}{x}$ is defined), $\frac{1}{x} > 0$ (in order $\log \frac{1}{x}$ is defined) and $\log \frac{1}{x} > 0$ (in order $\log \log \frac{1}{x}$ is defined). If we put it together, we get that the domain of f is $(0, 1)$.

The domain of f' is the same, as no more problems appear.

The formula for f' can be computed by standard rules, so

$$f'(x) = \exp(\sin x \cdot \log \log \frac{1}{x}) \cdot \left(\cos x \cdot \log \log \frac{1}{x} + \sin x \cdot \frac{1}{\log \frac{1}{x}} \cdot \frac{1}{x} \cdot \left(-\frac{1}{x^2} \right) \right),$$

which can be, of course, a bit simplified.

Evaluation: Two students computed the correct formula for f' . One of them did not take care of the domain (this one obtained 0.6 points), the other one took care, but the answer was not complete (this one obtained 0.7 points).

Three students tried to compute the formula for f' in a correct way, but they omitted something. Two of them did not take care of the domain (they obtained 0.4 points), the third one took care, but the answer was not complete (this one obtained 0.5 points).

The remaining seven students tried to compute the formula for f' in a completely wrong way. They obtained 0 points.