TEST3, KRUH 4, PŘÍKLAD A1

$$\int_0^{\pi/6} \frac{dx}{\cos^3 x} = \int_0^{\pi/6} \frac{\cos x}{\cos^4 x}\,dx = \int_0^{\pi/6} \frac{\cos x}{(1-\sin^2 x)^2}\,dx = \quad \text{1 bod}$$

substituce $y = \sin x$ (1 bod), $x \in (0, \frac{\pi}{6}) \Rightarrow y \in (0, \frac{1}{2})$, $\frac{dy}{dx} = \cos x > 0$ (1 bod)

$$= \int_0^{1/2} \frac{dy}{(1-y^2)^2} = \int_0^{1/2} \frac{dy}{(1-y)^2(1+y)^2} = (*) \quad \text{1 bod}$$

$$\frac{1}{(1-y)^2(1+y)^2} = \frac{A}{y-1} + \frac{B}{(y-1)^2} + \frac{C}{(y+1)} + \frac{D}{(y+1)^2} \quad \text{1 bod}$$

2 body

$$1 = A(y-1)(y+1)^2 + B(y+1)^2 + C(y+1)(y-1)^2 + D(y-1)^2$$

$y=1$: $\quad 1 = B \cdot 4 \Rightarrow B = \frac{1}{4}$

$y=-1$: $\quad 1 = D \cdot 4 \Rightarrow D = \frac{1}{4}$

u y^3: $\quad 0 = A + C$

u y^0: $\quad 1 = -A + B + C + D$

$A = -C$

$\frac{1}{2} = -A + C = 2C$

$\Rightarrow C = \frac{1}{4}, \; A = -\frac{1}{4}$

$$(*) = \int_0^{1/2} \left(\frac{-\frac{1}{4}}{y-1} + \frac{\frac{1}{4}}{(y-1)^2} + \frac{\frac{1}{4}}{y+1} + \frac{\frac{1}{4}}{(y+1)^2} \right) dy =$$

$$= \left[-\frac{1}{4}\lg|y-1| - \frac{\frac{1}{4}}{y-1} + \frac{1}{4}\lg|y+1| - \frac{\frac{1}{4}}{y+1} \right]_0^{1/2} = \quad \text{2 body}$$

$$= -\frac{1}{4}\lg\frac{1}{2} - \frac{\frac{1}{4}}{-\frac{1}{2}} + \frac{1}{4}\lg\frac{3}{2} - \frac{\frac{1}{4}}{\frac{3}{2}} + 0 + \frac{\frac{1}{4}}{-1} - 0 + \frac{\frac{1}{4}}{1} =$$

$$\frac{1}{4}\left(\lg 2 + \lg\frac{3}{2}\right) + \frac{1}{2} - \frac{1}{6} - \frac{1}{4} + \frac{1}{4} = \frac{1}{4}\lg 3 + \frac{1}{3} \quad \text{1 bod}$$

PŘÍKLAD B1

$$\int_{\pi/6}^{\pi/2} \frac{dx}{\sin^3 x} = \int_{\pi/6}^{\pi/2} \frac{\sin x}{\sin^4 x}\,dx = \int_{\pi/6}^{\pi/2} \frac{\sin x}{(1-\cos^2 x)}\,dx = (*)$$

Substituce $y = \cos x$, $x \in (\frac{\pi}{6}, \frac{\pi}{2}) \Rightarrow y \in (0, \frac{\sqrt{3}}{2})$, $\frac{dy}{dx} = -\sin x \leq 0$

$$(*) = \int_0^{\sqrt{3}/2} \frac{1}{(1-y^2)^2}\,dy = \text{jako v (A)}$$

$$\ldots = \cancel{\left[-\frac{1}{4}\frac{1}{y-1} + \frac{1}{4} \right] \left[-\frac{1}{4} \lg|y-1| - \frac{1}{4}\frac{1}{y-1} \right]}$$

PŘÍKLAD 42

$$\int_0^1 \frac{e^x - \cos x - x - x^2}{x^{5/2}\sqrt{1-x^2}}\,dx$$

1 bod

funkce je spojitá na $(0,1)$, stačí vyšetřit u $0+$ a u $1-$

u $0+$: použiju Taylorův polynom

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + o(t^3)$$
$$\cos t = 1 - \frac{t^2}{2} + o(t^3)$$
$$\Rightarrow e^t - \cos t - t - t^2 = \frac{t^3}{6} + o(t^3) \quad \text{pro } t \to 0$$

2 body

Protože $\sqrt{1-t^2} \to 1$ pro $t \to 0+$, srovnáme s $\frac{t^3}{t^{5/2}} = \sqrt{t}$

$$\lim_{t\to 0+} \frac{\frac{e^t - \cos t - t - t^2}{t^{5/2}\sqrt{1-t^2}}}{\frac{t^3}{t^{5/2}}} = \lim_{t\to 0} \frac{e^t - \cos t - t - t^2}{t^3\sqrt{1-t^2}} =$$
$$= \lim_{t\to 0} \frac{\frac{t^3}{6} + o(t^3)}{t^3\sqrt{1-t^2}} = \frac{1}{6}$$

1 bod

Tedy $\int_0^{1/2}$ konv. $\Leftrightarrow$ konv. $\int_0^{1/2} \frac{t^3}{t^{5/2}} = \int_0^{1/2} \sqrt{t}\,dt$, a ta konverguje,

Závěr u $0+$: $\int_0^{1/2}$ konverguje (absolutně)

2 body

u $1-$: srovnáme s $\frac{1}{\sqrt{1-x}}$

3 body

$$\lim_{t\to 1-} \frac{\frac{e^t - \cos t - t - t^2}{t^{5/2}\sqrt{1-t^2}}}{\frac{1}{\sqrt{1-t}}} = \lim_{t\to 1-} \frac{e^t - \cos t - t - t^2}{t^{5/2}\sqrt{1+t}} = \frac{e - \cos 1 - 2}{\sqrt{2}} \in \mathbb{R}$$

Protože $\int_{1/2}^1 \frac{1}{\sqrt{1-x}}\,dx$ konv., konv. i náš integrál!

Závěr: Integrál konverguje absolutně

1 bod

PŘÍKLAD B2

stejně jako A2 : fce spojitá na $(-1,0)$

v $-1+$ srovnat s $\frac{1}{\sqrt{1+x}}$

v $0-$ srovnat s $\sqrt[3]{x}$

tedy konverguje absolutně

PŘÍKLAD A3

$f(x,y) = (x+y)\sin(|x|-|y|)$

$D_f = \mathbb{R}^2$ 1 bod

$\frac{\partial f}{\partial x}(x,y) = \sin(|x|-|y|) + (x+y)\cdot\cos(|x|-|y|)\cdot \operatorname{sgn} x$ 1 bod

pro $x \neq 0$, tj. mimo osu y 1 bod

$\frac{\partial f}{\partial y}(x,y) = \sin(|x|-|y|) + (x+y)\cdot\cos(|x|-|y|)\cdot(-\operatorname{sgn} y)$ 1 bod

pro $y \neq 0$, tj. mimo osu x 1 bod

3 body

$\frac{\partial f}{\partial x}(0,y) \underset{\nearrow}{=} \lim_{x\to 0} \frac{\partial f}{\partial x}(x,y) = \lim_{x\to 0} \sin(|x|-|y|) + (x+y)\cos(|x|-|y|)\operatorname{sgn} x$

$x \mapsto f(x,y)$ spojitá

$= \begin{cases} -\sin|y| + y\cos|y| & \text{zprava} \\ -\sin|y| - y\cos|y| & \text{zleva} \end{cases}$

oboustranná limita existuje $\iff y\cos|y| = 0$, tj. $y = 0$ nebo $y = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

2 body

$\frac{\partial f}{\partial y}(x,0) \underset{\nearrow}{=} \lim_{y\to 0} \frac{\partial f}{\partial y}(x,y) = \lim_{y\to 0} \sin(|x|-|y|) + (x+y)\cos(|x|-|y|)\cdot(-\operatorname{sgn} y) =$

$y \mapsto f(x,y)$ spojitá

$= \begin{cases} \sin|x| + x\cos|x| & \text{zprava} \\ \sin x + x\cos|x| & \text{zleva} \end{cases}$

opět oboustranná limita ex. $\iff x\cos|x| = 0$, tj. $x = 0$ nebo $x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

Závěr $\frac{\partial f}{\partial x}(x,y) =$ viz výše pro $x \neq 0$

$\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial x}(0, \frac{\pi}{2}+k\pi) = -\sin|\frac{\pi}{2}+k\pi| = \begin{cases} (-1)^{k+1} & k \geq 0 \\ (-1)^k & k < 0 \end{cases}$ $k \in \mathbb{Z}$

$\frac{\partial f}{\partial y}(x,y) =$ viz výše pro $y \neq 0$

$\frac{\partial f}{\partial y}(0,0) = 0$, $\frac{\partial f}{\partial y}(\frac{\pi}{2}+k\pi, 0) = \begin{cases} (-1)^{k+1} & k \geq 0 \\ (-1)^k & k < 0 \end{cases}$ $k \in \mathbb{Z}$

v ostatních bodech příslušné p.d. neex.

PŘÍKLAD B3

$f(x,y) = (x-y)\cos(|x|+|y|)$

$D_f = \mathbb{R}^2$

$\frac{\partial f}{\partial x}(x,y) = \cos(|x|+|y|) + (x-y)\cdot(-\sin(|x|+|y|))\cdot \operatorname{sgn}(x)$

pro $x \neq 0$, tj. mimo osu y

$\frac{\partial f}{\partial y}(x,y) = -\cos(|x|+|y|) + (x-y)\cdot(-\sin(|x|+|y|))\cdot \operatorname{sgn}(y)$

pro $y \neq 0$, tj. mimo osu x

$$\frac{\partial f}{\partial x}(0,y) = \lim_{x\to 0} \frac{\partial f}{\partial x}(x,y) = \begin{cases} \cos|y| + y\sin|y| & \text{zprava} \\ \cos|y| - y\sin|y| & \text{zleva} \end{cases}$$

$x \mapsto f(x,y)$ spojitá

oboustranná derivace ex. $\iff y\sin|y| = 0$, tj. $y = k\pi,\ k\in\mathbb{Z}$

$$\frac{\partial f}{\partial x}(0,k\pi) = \cos|k\pi| = (-1)^k$$

$$\frac{\partial f}{\partial y}(x,0) = \lim_{y\to 0} \frac{\partial f}{\partial y}(x,y) = \begin{cases} -\cos|x| - x\sin|x| & \text{zprava} \\ -\cos|x| + x\sin|x| & \text{zleva} \end{cases}$$

$y \mapsto f(x,y)$ spojitá

oboustranná ex. $\iff x\sin|x| = 0$, tj. $x = k\pi,\ k\in\mathbb{Z}$

$$\frac{\partial f}{\partial y}(k\pi,0) = -\cos|k\pi| = (-1)^{k+1}$$