TEST 2, KRUH 3, PŘÍKLAD A1

$\int \sqrt{x^2-9}\, dx$ na $(-\infty,-3)$ ~~1 bod~~

1 bod

Eulerova substituce $y = \sqrt{x^2-9} - x$, $y \in (3,+\infty)$ 1 bod

$\lim_{x\to-\infty} \sqrt{x^2-9} - x = +\infty$

$\lim_{x\to-3^-} \sqrt{x^2-9} - x = 3$

$\sqrt{x^2-9} = x+y$

$x^2-9 = x^2+2xy+y^2$

$2xy = -9-y^2$

$x = \frac{-9-y^2}{2y} = -\frac{9}{2y} - \frac{y}{2}$

2 body

$\frac{dx}{dy} = \frac{9}{2y^2} - \frac{1}{2} < 0$ pro $y \in (3,+\infty)$ 1 bod

Počítáme tedy:

1 bod

$$\int \left(y - \frac{9}{2y} - \frac{y}{2}\right)\cdot\left(\frac{9}{2y^2} - \frac{1}{2}\right) dy = \int \left(\frac{y}{2} - \frac{9}{2y}\right)\left(\frac{9}{2y^2} - \frac{1}{2}\right) dy =$$

$$= \int \left(\frac{9}{4y} - \frac{y}{4} - \frac{81}{4y^3} + \frac{9}{4y}\right) dy \int \left(\frac{9}{2y} - \frac{y}{4} - \frac{81}{4y^3}\right) dy =$$

$$= \frac{9}{2}\ln y - \frac{y^2}{8} + \frac{81}{8y^2} + C \quad \text{na } (3,+\infty)$$

3 body

Tedy: $\int \sqrt{x^2-9}\, dx = \frac{9}{2}\ln\left(\sqrt{x^2-9} - x\right) - \frac{1}{8}\left(\sqrt{x^2-9} - x\right)^2 + \frac{81}{8\left(\sqrt{x^2-9}-x\right)^2} + C$

na $(-\infty,-3)$ 1 bod

PŘÍKLAD A1 – ALTERNATIVA

$$\int \sqrt{x^2-9}\, dx \qquad x \in (-\infty, -3)$$

$x = -3\cosh t$, $t \in (0, \infty)$ 1 bod $\qquad t = \operatorname{argcosh}\left(-\frac{x}{3}\right)$ 1 bod

$\frac{dx}{dt} = -3\sinh t < 0$ na $(0, \infty)$ 1 bod

počítáme na $(0,\infty)$:

$$\int \sqrt{9\cosh^2 t - 9}\,(-3\sinh t)\, dt = \int \sqrt{9\sinh^2 t}\cdot(-3\sinh t)\, dt = -9\int \sinh^2 t\, dt$$

1 bod $\qquad$ 2 body

2 body

$$\int \sinh^2 t\, dt = \int \sinh t \cdot \sinh t\, dt = \cosh t \sinh t - \int \cosh^2 t\, dt =$$

$$= \cosh t \sinh t - \int 1 - \int \sinh^2 t\, dt = \cosh t \sinh t - t - \int \sinh^2 t\, dt$$

$$\int \sinh^2 t\, dt = \frac{1}{2}(\cosh t \sinh t - t) + C$$

2 body

$$\Rightarrow \int \sqrt{x^2-9}\, dx = \frac{1}{2}\Big(\cosh\left(\operatorname{argcosh}\left(-\tfrac{x}{3}\right)\right)\sinh\left(\operatorname{argcosh}\left(-\tfrac{x}{3}\right)\right)$$

2 body

$$- \operatorname{argcosh}\left(-\tfrac{x}{3}\right)\Big) + C$$

na $(-\infty, -3)$

PŘÍKLAD B1 – VARIANTA ŘEŠENÍ

$$\int \sqrt{x^2-4}\,dx = \int \sqrt{\frac{x-2}{x+2}}\cdot(x+2)\,dx$$

na $(2,\infty)$ 1bod

$y = \sqrt{\frac{x-2}{x+2}}$ 1bod $x \in (2,\infty) \Rightarrow y \in (0,1)$ 1bod

$y^2 = \frac{x-2}{x+2}$

$y^2 \cdot x + 2y^2 = x - 2$

$x(y^2-1) = -2-2y^2$

$x = 2\cdot\frac{1+y^2}{1-y^2}$ 1bod

$$\frac{dx}{dy} = 2\cdot\frac{2y(1-y^2)-(1+y^2)\cdot(-2y)}{(1-y^2)^2} = \frac{8y}{(1-y^2)^2}$$ 1bod

Počítáme tedy: $\int y\cdot\left(2\cdot\frac{1+y^2}{1-y^2}+2\right)\cdot\frac{8y}{(1-y^2)^2}\,dy = \int\frac{32y^2}{(1-y^2)^3}$

1bod

parciální zlomky:

$$\frac{32y^2}{(1-y^2)^3} = \frac{A}{1-y}+\frac{B}{(1-y)^2}+\frac{C}{(1-y)^3}+\frac{D}{1+y}+\frac{E}{(1+y)^2}+\frac{F}{(1+y)^3}$$

2body

$$32y^2 = A(1-y)^2(1+y)^3 + B(1-y)(1+y)^3 + C(1+y)^3 + D(1-y)^3(1+y)^2 + E(1-y)^3(1+y) + F(1-y)^3$$

$y=1$: $32 = 8C \Rightarrow C = 4$

$y=-1$: $32 = 8F \Rightarrow F = 4$

$$32y^2 = A(1-y^2)^2(1+y) + B(1-y^2)(1+y)^2 + 4\cdot(1+y)^3 + D(1-y^2)^2(1-y) + E(1-y^2)(1-y)^2 + 4(1-y)^3$$

$$32y^2 = A(1-2y^2+y^4)(1+y) + B(1-y^2)(1+2y+y^2) + D(1-2y^2+y^4)(1-y) + E(1-y^2)(1-2y+y^2) + 8 + 24y^2$$

u y^5: $0 = A - D$

u y^4: $0 = A - B + D - E$

u y^3: $0 = -2A - 2B + 2D + 2E$

u y^2: $32 = -2A - 2D + 24$

$A = D$, $-4A = 8$ $\Rightarrow A = D = -2$

$B + E = -4$, $B = E$ $\Rightarrow B = E = -2$

$$\int\left(-\frac{2}{1-y}-\frac{2}{(1-y)^2}+\frac{4}{(1-y)^3}-\frac{2}{1+y}-\frac{2}{(1+y)^2}+\frac{4}{(1+y)^3}\right)dy=$$

1bod

$$=2\ln(1-y)-\frac{2}{1-y}+\frac{2}{(1-y)^2}-2\ln(1+y)+\frac{2}{1+y}-\frac{2}{(1+y)^2}+C$$

na $(0,1)$

Teď přim. fci

$$\int\sqrt{t^2-4}\,dt=2\cdot\ln\frac{1-\sqrt{\frac{t-2}{t+2}}}{1+\sqrt{\frac{x-2}{t+2}}}-\frac{2}{1-\sqrt{\frac{x-2}{t+2}}}+\frac{2}{1+\sqrt{\frac{t-2}{t+2}}}$$

$$+\frac{2}{\left(1-\sqrt{\frac{t-2}{t+2}}\right)^2}-\frac{2}{\left(1+\sqrt{\frac{t-2}{t+2}}\right)^2}+C \quad \text{na } (2,\infty)$$

1bod

TEST 2, PŘÍKLAD A2, KRUH 3

$$\int \frac{3x+4}{x^2+5x+7}\,dx = \int\left(\frac{3}{2}\,\frac{2x+5}{x^2+5x+7} + \frac{4-5\cdot\frac{3}{2}}{x^2+5x+7}\right)dx =$$ 2 body

(na $\mathbb{R}$, protože $5^2-4\cdot7<0$)

$$= \underbrace{\tfrac{3}{2}\lg(x^2+5x+7)}_{1\text{ bod}} + \underbrace{\int \frac{-\frac{7}{2}}{x^2+5x+7}\,dx}\ = (*)$$

$$= \int \frac{-\frac{7}{2}}{x^2+2x\cdot\frac{5}{2}+\frac{25}{4}+\frac{3}{4}} = \underbrace{\int \frac{-\frac{7}{2}}{(x+\frac{5}{2})^2+\frac{3}{4}}\,dx}_{1\text{ body}} = \underbrace{\int \frac{-\frac{7}{2}}{\frac{3}{4}\left(\frac{4}{3}(x+\frac{5}{2})^2+1\right)}\,dx}_{1\text{ body}} =$$

1 bod

$$= \int -\frac{14}{3}\,\frac{\sqrt{3}}{2}\,\frac{\frac{2}{\sqrt{3}}}{\left(\frac{2}{\sqrt{3}}(x+\frac{5}{2})\right)^2+1}\,dx = -\frac{7}{\sqrt{3}}\operatorname{arctg}\left(\frac{2}{\sqrt{3}}(x+\frac{5}{2})\right)+C$$

tj

$$(*) = \tfrac{3}{2}\lg(x^2+5x+7) - \frac{7}{\sqrt{3}}\operatorname{arctg}\frac{2x+5}{\sqrt{3}} + C$$ 1 b; 1 bod na $\mathbb{R}$

PŘÍKLAD B2

$$\int \frac{3x+4}{x^2-5x+7}\,dx = \int\left(\frac{3}{2}\,\frac{2x-5}{x^2-5x+7} + \frac{4+5\cdot\frac{3}{2}}{x^2-5x+7}\right)dx =$$

(na $\mathbb{R}$, protože $(-5)^2-4\cdot7<0$)

$$= \tfrac{3}{2}\lg(x^2-5x+7) + \underbrace{\int \frac{\frac{23}{2}}{x^2-5x+7}\,dx} = \tfrac{3}{2}\lg(x^2-5x+7) + \frac{23}{\sqrt{3}}\operatorname{arctg}\frac{2x-5}{\sqrt{3}} + C$$

na $\mathbb{R}$

$$\left(= \int \frac{\frac{23}{2}}{x^2-2x\cdot\frac{5}{2}+\frac{25}{4}+\frac{3}{4}}\,dx = \int \frac{\frac{23}{2}}{(x-\frac{5}{2})^2+\frac{3}{4}}\,dx = \int \frac{\frac{23}{2}}{\frac{3}{4}\left(\frac{4}{3}(x-\frac{5}{2})^2+1\right)}\,dx =\right.$$

$$\left.= \int \frac{46}{3}\cdot\frac{\sqrt{3}}{2}\,\frac{\frac{2}{\sqrt{3}}}{\left(\frac{2}{\sqrt{3}}(x-\frac{5}{2})\right)^2+1}\,dx = \frac{23}{\sqrt{3}}\operatorname{arctg}\left(\frac{2}{\sqrt{3}}(x-\frac{5}{2})\right)+C\right)$$

TEST 2, KRUH 3, PŘÍKLAD A3

$$\sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n-1} =: f(x) \qquad \lim_{n\to\infty} \sqrt[2n+1]{\frac{1}{2n-1}} = 1 \Rightarrow$$ pol. konv. je 1, počítám na $(-1,1)$ 1 bod

$$\parallel$$

$$x^2 \cdot \underbrace{\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}}_{g(x)}$$ 1 bod

$$g(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}, \; x \in (-1,1)$$

$$g'(x) = \sum_{n=1}^{\infty} x^{2n-2} = \frac{1}{1-x^2}, \; x \in (-1,1)$$ 3 body

3 body $$\int \frac{1}{1-x^2}\,dx = \int \left(\frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}\right) dx = -\frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) + C$$ na $(-1,1)$

1 bod $$g(0) = 0 \Rightarrow g(x) = \frac{1}{2}\ln\frac{1+x}{1-x}, \; x \in (-1,1)$$

$$\Rightarrow f(x) = \frac{x^2}{2}\ln\frac{1+x}{1-x}, \; x \in (-1,1)$$ 1 bod

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Příklad B3 $$\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n+1} =: f(x) \qquad \lim_{n\to\infty} \sqrt[2n-1]{\frac{1}{2n+1}} = 1$$ 1 bod

$$g(x) := x^2 \cdot f(x) = \sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1}, \; x \in (-1,1)$$ 1 bod

$$g'(x) = \sum_{n=1}^{\infty} x^{2n} = \frac{x^2}{1-x^2}, \; x \in (-1,1)$$ 3 body

$$\int \frac{x^2}{1-x^2}\,dx = \int -1 + \frac{1}{1-x^2}\,dx = -x + \int\left(\frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}\right)dx =$$ 3 body

$$= -x + \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x) + C \text{ na } (-1,1)$$

$$g(0) = 0 \Rightarrow g(x) = -x + \frac{1}{2}\ln\frac{1+x}{1-x}, \; x \in (-1,1)$$ 1 bod

$$\Rightarrow f(x) = \begin{cases} \dfrac{-x + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}{x^2}, & x \in (-1,0) \cup (0,1) \\ 0 & x = 0 \end{cases}$$ [toponechávám na vás] 1 bod