III. Determine sup and inf of the function f on the set M and decide whether these values are attained

1. $f(x,y,z) = (x+y)^2 + (x-y)^2 + z$; $M = [-1;1] \times [-1;1] \times [-1;1]$;
2. $f(x,y) = x^2 - xy + y^2$, $\quad M = \{[x,y] \in \mathbb{R}^2;\ |x| + |y| \le 1\}$
3. $f(x,y) = \frac{x}{a} + \frac{y}{b}, a > 0, b > 0$; $M = \{[x,y]; x^2 + y^2 \le 1\}$;
4. $f(x,y) = \left(x^2 + y^2\right) e^{-\left(x^2+y^2\right)}$; $M = \mathbb{R}^2$; **5.** $f(x,y) = \left(x^2 + 5y^2\right) e^{-\left(3x^2+y^2\right)}$; $M = \mathbb{R}^2$;
6. $f(x,y) = (x+y)\, e^{-2x-3y}$; $M = \{[x,y]; x > 0, y > 0\}$
7. $f(x,y,z) = x^2 + y^2 + z^2$; $M = \{[x,y,z]; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$; $a > b > c > 0$.

Answers and hints. **1.** max 5 at points $[1,1,1]$, $[1,-1,1]$, $[-1,1,1]$, $[-1,-1,1]$, min -1 at the point $[0,0,-1]$. (Simplify the formula for f and show the result without computing derivatives.) **2.** max 1 at points $[\pm1,0]$, $[0,\pm1]$, min 0 at the point $[0,0]$. **3.** max $\frac{\sqrt{a^2+b^2}}{ab}$ at the point $[\frac{b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}]$, min $-\frac{\sqrt{a^2+b^2}}{ab}$ at the point $[-\frac{b}{\sqrt{a^2+b^2}}, -\frac{a}{\sqrt{a^2+b^2}}]$. (Distinguish the interior and the boundary. For the boundary circle, use the parametrization $x = \cos t$, $y = \sin t$.) **4.** min 0 at the point $[0,0]$, max $\frac{1}{e}$ at points of the circle $x^2+y^2 = 1$. (Use the behaviour of the function $t \mapsto te^{-t}$.) **5.** max $\frac{5}{e}$ at the points $[0,\pm1]$, min 0 at the point $[0,0]$. (The value of min is obvious, to show that max is attained use that f is bounded from above by $5(3x^2+y^2)e^{-(3x^2+y^2)}$.) **6.** sup $\frac{1}{2e}$, not attained; inf 0, not attained. (Use the fact that sup and inf of f on M is the same as on $\overline{M}$. Show that on $\overline{M}$ the respective values are attained using similar ideas as in the previous problem.) **7.** max a^2 v bodech $[\pm a, 0, 0]$, min 0 v $[0,0,0]$. (Use geometrical interpretation to determine the values without computing derivatives.)