5.2. **Invertible matrices and rank of a matrix.**

**Definition.** Let $\mathbb{A} \in M(n\times n)$. We say that $\mathbb{A}$ is an *invertible* matrix, if there exists $\mathbb{B} \in M(n\times n)$ such that

$$\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A} = \mathbb{I}.$$

**Definition.** We say that $\mathbb{B} \in M(n\times n)$ is an *inverse* of a matrix $\mathbb{A} \in M(n\times n)$, if $\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A} = \mathbb{I}$.

*Remark.*

- A matrix $\mathbb{A} \in M(n \times n)$ is invertible, if and only if it has an inverse.
- Each matrix $\mathbb{A} \in M(n \times n)$ has at most one inverse. If it exists, it is denoted by $\mathbb{A}^{-1}$.
- If $\mathbb{A}, \mathbb{B} \in M(n \times n)$ are such that $\mathbb{A}\mathbb{B} = \mathbb{I}$, then also $\mathbb{B}\mathbb{A} = \mathbb{I}$, hence $\mathbb{B} = \mathbb{A}^{-1}$. (This is not obvious, it follows from the section 5.5)

**Theorem 5.4** (invertibity and matrix operations)**.** *Let $\mathbb{A}, \mathbb{B} \in M(n \times n)$ be invertible. Then we have:*

*(i)* $\mathbb{A}^{-1}$ *is invertible and* $\left(\mathbb{A}^{-1}\right)^{-1} = \mathbb{A}$,
*(ii)* $\mathbb{A}^T$ *is invertible and* $\left(\mathbb{A}^T\right)^{-1} = \left(\mathbb{A}^{-1}\right)^T$,
*(iii)* $\mathbb{A}\mathbb{B}$ *is invertible and* $(\mathbb{A}\mathbb{B})^{-1} = \mathbb{B}^{-1}\mathbb{A}^{-1}$.

**Definition.** Let $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k \in \mathbf{R}^n$ be vectors. *Linear combination* of vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ is an expression $\lambda_1\boldsymbol{v}^1 + \cdots + \lambda_k\boldsymbol{v}^k$, where $\lambda_1, \dots, \lambda_k \in \mathbf{R}$. By *trivial linear combination* of vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ we mean the linear combination $0 \cdot \boldsymbol{v}^1 + \cdots + 0 \cdot \boldsymbol{v}^k$. Linear combination, which is not trivial, is called *nontrivial*.

**Definition.** We say that vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ are *linearly dependent*, if there exists their nontrivial linear combination, which is equal to the zero vector.

We say that vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ are *linearly independent*, if they are not linearly dependent, i.e., if whenever $\lambda_1, \dots, \lambda_k \in \mathbf{R}$ satisfy $\lambda_1\boldsymbol{v}^1 + \cdots + \lambda_k\boldsymbol{v}^k = \boldsymbol{o}$, then $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$.

**Definition.** Let $\mathbb{A} \in M(m \times n)$. *Rank* of the matrix $\mathbb{A}$ is the maximal number of linearly independent row vectors of $\mathbb{A}$. Rank of $\mathbb{A}$ is denoted by $\operatorname{rk}(\mathbb{A})$.

*Remark.* $\operatorname{rk}(\mathbb{A}) = k$ means that there is a $k$-tuple of rows which is linearly independent and that any $(k+1)$-tuple of rows is linearly dependent.

**Definition.** We say that $\mathbb{A} \in M(m \times n)$ is in the *row echelon form*, if for each $i \in \{2, \dots, m\}$ we have, that either the $i$th row of $\mathbb{A}$ is a zero vector or the number of zeros at the beginning of the $i$th row is strictly bigger than the number of zeros at the beginning of $(i-1)$th row.

*Remark.* The rank of a row echelon matrix $\mathbb{A}$ is equal to the number of nonzero rows of $\mathbb{A}$.

**Definition.** *Elementary row transformations* of the matrix $\mathbb{A}$ are defined as:

(i) interchange of two rows,
(ii) multiplication of a row by a nonzero real number,
(iii) addition of a multiple of a row to another row.

**Definition.** *Transformation* is defined as a finite sequence of elementary row transformations. If the matrix $\mathbb{B} \in M(m \times n)$ was created from $\mathbb{A} \in M(m \times n)$ applying a transformation $T$ to $\mathbb{A}$, then this fact is denoted by $\mathbb{A} \overset{T}{\rightsquigarrow} \mathbb{B}$.

**Theorem 5.5** (properties of transformation)**.**

*(i)* *Let* $\mathbb{A} \in M(m \times n)$. *Then there exists a transformation transforming* $\mathbb{A}$ *to a row echelon matrix.*

*(ii)* *Let* $T_1$ *be a transformation applicable to* $m$*-by-*$n$ *matrices. Then there exists a transformation* $T_2$ *applicable to* $m$*-by-*$n$ *matrices such that if* $\mathbb{A} \overset{T_1}{\leadsto} \mathbb{B}$ *for some* $\mathbb{A}, \mathbb{B} \in M(m \times n)$, *then* $\mathbb{B} \overset{T_2}{\leadsto} \mathbb{A}$.

*(iii)* *Let* $\mathbb{A}, \mathbb{B} \in M(m \times n)$ *and there exist a transformation* $T$ *such that* $\mathbb{A} \overset{T}{\leadsto} \mathbb{B}$. *Then* $\mathrm{rk}(\mathbb{A}) = \mathrm{rk}(\mathbb{B})$.

**Theorem 5.6** (multiplication and transformation)**.** *Let* $\mathbb{A} \in M(m \times k)$, $\mathbb{B} \in M(k \times n)$, $\mathbb{C} \in M(m \times n)$ *and we have* $\mathbb{A}\mathbb{B} = \mathbb{C}$. *Let* $T$ *be a transformation and* $\mathbb{A} \overset{T}{\leadsto} \mathbb{A}'$ *and* $\mathbb{C} \overset{T}{\leadsto} \mathbb{C}'$. *Then we have* $\mathbb{A}'\mathbb{B} = \mathbb{C}'$.

**Lemma 5.7.** *Let* $\mathbb{A} \in M(n \times n)$ *and* $\mathrm{rk}(\mathbb{A}) = n$. *Then there exists a transformation transforming* $\mathbb{A}$ *to* $\mathbb{I}$.

**Theorem 5.8.** *Let* $\mathbb{A} \in M(n \times n)$. *Then* $\mathbb{A}$ *is invertible if and only if* $\mathrm{rk}(\mathbb{A}) = n$.

*Remark.*

- Similarly as elementary row transformations one can define *elementary column transformations*. A finite sequence of elementary column transformations is then called a *column transformation*.
- It is not hard to check that a column transformation does not change the rank of a matrix.
- Using this fact it is easy to deduce that $\mathrm{rk}(\mathbb{A}) = \mathrm{rk}(\mathbb{A}^T)$ for any matrix $\mathbb{A}$. (Let us transform $\mathbb{A}$ to a row echelon matrix $\mathbb{B}$ by a transformation. Then $\mathrm{rk}(\mathbb{A}) = \mathrm{rk}(\mathbb{B})$ and, moreover, by the previous item, $\mathrm{rk}(\mathbb{A}^T) = \mathrm{rk}(\mathbb{B}^T)$. Finally, it is not hard to check that $\mathrm{rk}(\mathbb{B}) = \mathrm{rk}(\mathbb{B}^T)$.)