4.3. **Continuous functions of several variables.**

Definition. Let $M \subset \mathbf{R}^n$, $\boldsymbol{x} \in M$, and $f\colon M \to \mathbf{R}$. We say that f is *continuous at $\boldsymbol{x}$ with respect to M*, if we have

$$\forall \varepsilon \in \mathbf{R}, \varepsilon > 0\ \exists \delta \in \mathbf{R}, \delta > 0\ \forall \boldsymbol{y} \in B(\boldsymbol{x}, \delta) \cap M\colon f(\boldsymbol{y}) \in B(f(\boldsymbol{x}), \varepsilon).$$

We say that f is *continuous at the point $\boldsymbol{x}$*, it it is continuous at $\boldsymbol{x}$ with respect to a neighborhood of $\boldsymbol{x}$, i.e.,

$$\forall \varepsilon \in \mathbf{R}, \varepsilon > 0\ \exists \delta \in \mathbf{R}, \delta > 0\ \forall \boldsymbol{y} \in B(\boldsymbol{x}, \delta)\colon f(\boldsymbol{y}) \in B(f(\boldsymbol{x}), \varepsilon).$$

Remark. Let $M \subset \mathbf{R}^n$, $\boldsymbol{x} \in M$, $f\colon M \to \mathbf{R}$, $g\colon M \to \mathbf{R}$, and $c \in \mathbf{R}$. If f and g are continuous at the point $\boldsymbol{x}$ with respect to M, then the functions cf, $f+g$ a fg are continuous at $\boldsymbol{x}$ with respect to M. If the function g is nonzero at each point of M, then also the function f/g is continuous at $\boldsymbol{x}$ with respect to M.

Theorem 4.7 (Heine). *Let $M \subset \mathbf{R}^n$, $\boldsymbol{x} \in M$, and $f\colon M \to \mathbf{R}$. Then the following are equivalent.*

(i) The function f is continuous at $\boldsymbol{x}$ with respect to M.
(ii) For each sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ such that $\boldsymbol{x}^j \in M$ pro $j \in \mathbf{N}$ a $\lim\limits_{j\to\infty} \boldsymbol{x}^j = \boldsymbol{x}$, we have $\lim\limits_{j\to\infty} f(\boldsymbol{x}^j) = f(\boldsymbol{x})$.

Remark. Let $r, s \in \mathbf{N}$, $M \subset \mathbf{R}^s$, $L \subset \mathbf{R}^r$, and $\boldsymbol{y} \in M$. Let $\varphi_1, \dots, \varphi_r$ be functions defined on M, which are continuous at $\boldsymbol{y}$ with respect to M and $[\varphi_1(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})] \in L$ for each $\boldsymbol{x} \in M$. Let $f\colon L \to \mathbf{R}$ be continuous at the point $[\varphi_1(\boldsymbol{y}), \dots, \varphi_r(\boldsymbol{y})]$ with respect to L. Then the composed function $F\colon M \to \mathbf{R}$ defined by

$$F(\boldsymbol{x}) = f\big(\varphi_1(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})\big), \quad \boldsymbol{x} \in M,$$

is continuous at $\boldsymbol{y}$ with respect to M.

Definition. Let $M \subset \mathbf{R}^n$ a $f\colon M \to \mathbf{R}$. We say that f is *continuous on M*, if it is continuous at each point $\boldsymbol{x} \in M$ with respect to M.

Remark. The projection $\pi_j\colon \mathbf{R}^n \to \mathbf{R}$, $\pi_j(\boldsymbol{x}) = x_j$, $1 \le j \le n$, are continuous on $\mathbf{R}^n$.

Theorem 4.8.

(1) Let f be a function continuous on an open set $G \subset \mathbf{R}^n$ and $c \in \mathbf{R}$. Then the set $\{\boldsymbol{x} \in G;\ f(\boldsymbol{x}) < c\}$ is open in $\mathbf{R}^n$.
(2) Let f be a function continuous on a closed set $F \subset \mathbf{R}^n$ and $c \in \mathbf{R}$. Then the set $\{\boldsymbol{x} \in F;\ f(\boldsymbol{x}) \le c\}$ is closed in $\mathbf{R}^n$.

Definition. We say that a function f of n variables has at a point $\boldsymbol{a} \in \mathbf{R}^n$ limit equal $A \in \mathbb{R}^*$, if we have

$$\forall \varepsilon \in \mathbf{R}, \varepsilon > 0\ \exists \delta \in \mathbf{R}, \delta > 0\ \forall \boldsymbol{x} \in B(\boldsymbol{a}, \delta) \setminus \{\boldsymbol{a}\}\colon f(\boldsymbol{x}) \in B(A, \varepsilon).$$

Remark.

- Each function has at a given point at most one limit. We write $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = A$.
- The function f is continuous at $\boldsymbol{a}$ if and only if $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = f(\boldsymbol{a})$.

- For functions of several variables one can prove similar theorems as for functions of one variable (arithmetics, sandwich theorem, …).

Theorem 4.9. *Let $r, s \in \mathbf{N}$, $\boldsymbol{a} \in \mathbf{R}^s$, and let $\varphi_1, \dots, \varphi_r$ be functions of s variables such that $\lim_{\boldsymbol{x}\to\boldsymbol{a}} \varphi_j(\boldsymbol{x}) = b_j$, $j = 1, \dots, r$. Set $\boldsymbol{b} = [b_1, \dots, b_r]$. Let f be a function of r variables whic is continuous at the point $\boldsymbol{b}$. We define a function F of s variables by*

$$F(\boldsymbol{x}) = f(\varphi_1(\boldsymbol{x}), \varphi_2(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})).$$

Then $\lim_{\boldsymbol{x}\to\boldsymbol{a}} F(\boldsymbol{x}) = f(\boldsymbol{b})$.

4.4. **Compact sets and their applications.**

Definition. We say that a set $M \subset \mathbf{R}^n$ is *compact*, if for each sequence of elements of M there exists a convergent subsequence with limit in M.
We say that a set $M \subset \mathbf{R}^n$ is *bounded*, if there exists $r > 0$ such that $M \subset B(\boldsymbol{o}, r)$.

Theorem 4.10 (characterization of compact subsets of $\mathbf{R}^n$)**.** *The set $M \subset \mathbf{R}^n$ is compact if and only if M is bounded and closed.*

Theorem 4.11 (attaining extrema)**.** *Let $M \subset \mathbf{R}^n$ be a nonempty compact set and $f\colon M \to \mathbf{R}$ be continuous on M. Then f attains on M its maximum and minimum.*

Corollary 4.12. *Let $M \subset \mathbf{R}^n$ be a nonempty compact set and $f\colon M \to \mathbf{R}$ be continuous on M. Then f is bounded on M.*