6. INTEGRALS

6.1. Riemann integral.

Definition. A finite sequence $\{x_j\}_{j=0}^n$ is called a *partition of the interval* $\langle a, b\rangle$, if we have

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The points $x_0, \ldots, x_n$ are called *partition points*.

By the *norm of partition* $D = \{x_j\}_{j=0}^n$ we mean

$$\nu(D) = \max\{x_j - x_{j-1};\ j = 1, \ldots, n\}.$$

We say that a partition D' of an interval $\langle a, b\rangle$ is a *refinement of the partition* D of the interval $\langle a, b\rangle$, if each point of D is a partition point of D'.

Definition. Let f be a bounded function on an interval $\langle a, b\rangle$.

If $D = \{x_j\}_{j=0}^n$ is a partition of $\langle a, b\rangle$, we denote

$$\overline{S}(f, D) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}),\ \text{where } M_j = \sup\{f(x);\ x \in \langle x_{j-1}, x_j\rangle\}$$

(the upper Darboux sum of f with respect to D),

$$\underline{S}(f, D) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}),\ \text{where } m_j = \inf\{f(x);\ x \in \langle x_{j-1}, x_j\rangle\}$$

(the lower Darboux sum of f with respect to D).

We further define

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf\big\{\overline{S}(f, D);\ D \text{ is a partition of the interval } \langle a, b\rangle\big\}$$

(the upper Riemann integral of f over $\langle a, b\rangle$),

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup\big\{\underline{S}(f, D);\ D \text{ is a partition of the interval } \langle a, b\rangle\big\}$$

(the lower Riemann integral of f over $\langle a, b\rangle$).

Definition. We say that a bounded function f is *Riemann-integrable* over the interval $\langle a, b\rangle$, if $\overline{\int_a^b} f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x$. Then the *Riemann integral* of f over the interval $\langle a, b\rangle$ is defined to be $\overline{\int_a^b} f(x)\,\mathrm{d}x$ and is denoted by $\int_a^b f(x)\,\mathrm{d}x$. If $a > b$, we define $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$. If $a = b$, we define $\int_a^b f(x)\,\mathrm{d}x = 0$.

Proposition 6.1 (basic properties)**.** *Let f be a bounded function on the interval $\langle a, b\rangle$.*

(i) Let D and D' be partitions of the interval $\langle a, b\rangle$ such that D' refines D. Then we have

$$\underline{S}(f, D) \le \underline{S}(f, D') \le \overline{S}(f, D') \le \overline{S}(f, D).$$

(ii) Let D_1 and D_2 be partitions of $\langle a, b\rangle$. Then there is a partition D which refines both D_1 and D_2.

(iii) Let D_1 and D_2 be partitions of $\langle a, b\rangle$. Then $\underline{S}(f, D_1) \le \overline{S}(f, D_2)$.

(iv) $\underline{\int_a^b} f(x)\,\mathrm{d}x \le \overline{\int_a^b} f(x)\,\mathrm{d}x$.

(v) f is Riemann-integrable over $\langle a, b\rangle$ if and only if for each $\varepsilon > 0$ there is a partition D with $\overline{S}(f, D) - \underline{S}(f, D) < \varepsilon$.

Theorem 6.2 (Riemann integral as an interval function). *(i) Let a function f be Riemann-integrable over $\langle a, b\rangle$ and let $\langle c, d\rangle \subset \langle a, b\rangle$. Then f is Riemann-integrable over $\langle c, d\rangle$.*

(ii) Let $c \in (a, b)$ and a function f is Riemann-integrable both over $\langle a, c\rangle$ and over $\langle c, b\rangle$. Then f is Riemann-integrable over $\langle a, b\rangle$ and we have

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

Theorem 6.3 (linearity of Riemann integral). *Let f and g be Riemann-integrable over $\langle a, b\rangle$ and let $\alpha \in \mathbf{R}$. Then*

(i) the function αf is Riemann-integrable over $\langle a, b\rangle$ and it holds

$$\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x,$$

(ii) the function $f + g$ is Riemann-integrable over $\langle a, b\rangle$ and it holds

$$\int_a^b \big(f(x) + g(x)\big)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x.$$

Theorem 6.4 (monotonicity of Riemann integral). *Let $a, b \in \mathbf{R}$, $a < b$, and let f and g be Riemann-integrable over $\langle a, b\rangle$.*

(i) If $f(x) \ge 0$ for each $x \in \langle a, b\rangle$, then

$$\int_a^b f(x)\,\mathrm{d}x \ge 0.$$

(ii) If $f(x) \le g(x)$ for each $x \in \langle a, b\rangle$, then

$$\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x.$$

(iii) The function $|f|$ is Riemann-integrable over $\langle a, b\rangle$ and it holds

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x.$$

Theorem 6.5 (existence of Riemann integral). *Let a function f be continuous on the interval $\langle a, b\rangle$, $a, b \in \mathbf{R}$. Then f is Riemann integrable over $\langle a, b\rangle$.*

Theorem 6.6 (differentiating indefinite integral). *Let f be a continuous function on $\langle a, b\rangle$ and let $c \in \langle a, b\rangle$. If we denote $F(x) = \int_c^x f(t)\,\mathrm{d}t$ for $x \in (a, b)$, then $F'(x) = f(x)$ for each $x \in (a, b)$.*