5.4. **Systems of linear equations.**

System of m equations with n unknowns:

$$\text{(S)} \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

Matrix notation

$$\mathbb{A}\boldsymbol{x} = \boldsymbol{b},$$

where $\mathbb{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \in M(m \times n)$, $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in M(m \times 1)$ a $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M(n \times 1)$.

Theorem 5.17 (on systems with square matrix)**.** *Let $\mathbb{A} \in M(n \times n)$. Then the following are equivalent.*

(i) The matrix $\mathbb{A}$ is invertible.
(ii) The system (S) *has for each $\boldsymbol{b}$ a unique solution.*
(iii) The system (S) *has for each $\boldsymbol{b}$ at least one solution.*

Remark. The previous theorem says the following:

- If $\mathbb{A}$ is invertible, then for each $\boldsymbol{b}$ the system (S) has a unique solution.
- If $\mathbb{A}$ is not invertible, then for some $\boldsymbol{b}$ the system (S) has no solution.

It can be moreover shown, that, provided $\mathbb{A}$ is not invertible, then for some $\boldsymbol{b}$ the system (S) has infinitely many solutions ($\boldsymbol{b} = \boldsymbol{o}$ works). This follows from the next section.

Theorem 5.18 (Cramer's rule)**.** *Let $\mathbb{A} \in M(n \times n)$ be an invertible matrix, $\boldsymbol{b} \in M(n \times 1)$, $\boldsymbol{x} \in M(n \times 1)$, and $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$. Then*

$$x_j = \frac{\begin{vmatrix} a_{11} & \dots & a_{1,j-1} & b_1 & a_{1,j+1} & \dots & a_{1n} \\ \vdots & & & \vdots & & & \vdots \\ a_{n1} & \dots & a_{n,j-1} & b_n & a_{n,j+1} & \dots & a_{nn} \end{vmatrix}}{\det \mathbb{A}}$$

for $j = 1, \dots, n$.

Definition. The matrix

$$(\mathbb{A}|\boldsymbol{b}) = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

is called *augmented matrix of the system* (S).

Gauss elimination method. Consider the system (S).

- Let T be a transformation of matrices with m rows. Suppose that $\mathbb{A} \overset{T}{\rightsquigarrow} \mathbb{A}'$ and $\boldsymbol{b} \overset{T}{\rightsquigarrow} \boldsymbol{b}'$. Then the system $\mathbb{A}'\boldsymbol{x} = \boldsymbol{b}'$ has the same set of solutions as the system (S).
- The augmented matrix of the system (S) can be transformed to a row echelon matrix $(\mathbb{A}'|\boldsymbol{b}')$. Thus solving the system (S) can be reduced to solving the system $\mathbb{A}'\boldsymbol{x} = \boldsymbol{b}'$, which is much simpler.
- One can use only the original version of the transformation, i.e., a finite sequence of elementary **row** transformations.

Theorem 5.19 (solvability of a linear system)**.** *The system* (S) *has a solution if and only if the matrix has the same rank as the extended matrix of the system.*

Remark. The system (S) has a solution if and only if the vector $\boldsymbol{b}$ can be expressed as a linear combination of the columns of the matrix $\mathbb{A}$.

5.5. **Matrices and linear mappings.**

Definition. We say that a mapping $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is *linear* if

(i) $\forall \boldsymbol{u}, \boldsymbol{v} \in \mathbf{R}^n\colon f(\boldsymbol{u}+\boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v})$,

(ii) $\forall \lambda \in \mathbf{R}\; \forall \boldsymbol{u} \in \mathbf{R}^n\colon f(\lambda \boldsymbol{u}) = \lambda f(\boldsymbol{u})$.

Definition. Let $i \in \{1, \dots, n\}$. The vector

$$\boldsymbol{e}^i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots\ i\text{-th coordinate}$$

is called i-th *canonical vector* of the space $\mathbf{R}^n$. The set $\{\boldsymbol{e}^1, \dots, \boldsymbol{e}^n\}$ of all canonical vectors in $\mathbf{R}^n$ is called *canonical basis of the space* $\mathbf{R}^n$.

The properties of canonical vectors:

(i) $\forall \boldsymbol{x} \in \mathbf{R}^n\; \exists \lambda_1, \dots, \lambda_n \in \mathbf{R}\colon \boldsymbol{x} = \lambda_1 \boldsymbol{e}^1 + \dots + \lambda_n \boldsymbol{e}^n$,

(ii) the vectors $\boldsymbol{e}^1, \dots, \boldsymbol{e}^n$ are linearly independent.

Theorem 5.20 (representation of linear mappings)**.** *The mapping $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is linear if and only if there exists a matrix $\mathbb{A} \in M(m \times n)$ such that*

$$\forall \boldsymbol{u} \in \mathbf{R}^n\colon f(\boldsymbol{u}) = \mathbb{A}\boldsymbol{u} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

Remark. The matrix $\mathbb{A}$ from the previous theorem is uniquely determined and is called the *representing matrix* of the linear mapping f.

Theorem 5.21 (representing matrix of a composition)**.** *Let $f\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear mapping represented by matrix $\mathbb{A} \in M(m \times n)$ a $g\colon \mathbf{R}^m \to \mathbf{R}^k$ be a linear mapping represented by a matrix $\mathbb{B} \in M(k \times m)$. Then the composed mapping $g \circ f\colon \mathbf{R}^n \to \mathbf{R}^k$ is linear and is represented by the matrix $\mathbb{B}\mathbb{A}$.*

Theorem 5.22. *Let a mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be linear. Then the following are equivalent.*

(i) The mapping f is a bijection (i.e., f is an injective mapping $\mathbf{R}^n$ onto $\mathbf{R}^n$).

(ii) The mapping f is an injective mapping.

(iii) The mapping f is a mapping $\mathbf{R}^n$ onto $\mathbf{R}^n$.