Věta IX.21 $f \in H(D)$, $p \in [1,\infty]$

Pak $f \in H^p \iff \exists\, g \in L^p(\mathbb{T}) : f = P[g]$ a $\forall n<0 : \hat{g}(n) = 0$

Důkaz $\Rightarrow$: $f \in H^p \Rightarrow$ vezmeme f^* z Věty 20.
Pak $f^* \in L^p(\mathbb{T})$ a $P[f^*] = f$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad , z \in \mathbb{D}$$

$L^p(\mathbb{T}) \subset L^1(\mathbb{T}) \Rightarrow$ Fourierovy koeficienty mají smysl
$H^p \subset H^1 \Rightarrow$ máme $f_r \xrightarrow{r \to 1-} f^*$ v normě L^1

$$m \in \mathbb{Z} \quad \widehat{f^*}(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(e^{it})\, e^{-imt}\, dt =$$

$$\underset{f_r \xrightarrow{L^1} f^*}{=} \lim_{r \to 1-} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{it})\, e^{-imt}\, dt =$$

$$= \lim_{r \to 1-} \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=0}^{\infty} a_n r^n e^{int} e^{-imt}\, dt =$$

$$= \lim_{r \to 1-} \sum_{n=0}^{\infty} a_n r^n \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)t}\, dt}_{\substack{= 0 \; n \neq m \\ = 1 \; n = m}} = \begin{cases} 0 & m < 0 \\ a_m & m \geq 0 \end{cases}$$

$\Leftarrow$ Nechť $g \in L^p(\mathbb{T})$, $\hat{g}(n) = 0$ pro $n < 0$

Položme $f = P[g]$

- f harmonická na $\mathbb{D}$ ($g \in L^1(\mathbb{T})$, lze použít Věta 3
- f holomorfní, protože $f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{g(w)}{w-z}\, dw$, $z \in \mathbb{D}$

(kde γ je kladně orientovaná jednotková kružnice)

$$\frac{1}{2\pi i}\int_\gamma \frac{g(w)}{w-z}\,dw - P[g](z) = \qquad (z = re^{i\theta})$$

$$= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{g(e^{it})}{e^{it}-re^{i\theta}}\, e^{it}\cdot i\,dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, g(e^{it})\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{it}) \underbrace{\left(\frac{e^{it}}{e^{it}-re^{i\theta}} - P_r(\theta-t)\right)}\,dt = (*)$$

$$= \frac{1}{1-re^{i(\theta-t)}} - P_r(\theta-t) = \sum_{n=0}^{\infty} r^n e^{in(\theta-t)} - \sum_{n\in\mathbb{Z}} r^{|n|} e^{in(\theta-t)}$$

$$= \sum_{n<0} r^{-n} e^{in(\theta-t)}$$

$$(*) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{it}) \cdot \sum_{n=+1}^{\infty} r^n e^{-in(\theta-t)}\,dt =$$

$$= \sum_{n=1}^{\infty} \frac{1}{2\pi} r^n e^{-in\theta} \int_{-\pi}^{\pi} g(e^{it})\, e^{int}\,dt = \sum_{n=1}^{\infty} r^n e^{-in\theta}\, \hat{g}(-n) = 0$$

• $f \in H^p$, można $\forall r \in (0,1)$ $M_p(f,r) \le \|g\|_p$

$p = \infty$ .. snadno

$p < \infty$ ... q sdružený exponent, $h \in L^q(\mathbb{T})$, $\|h\|_q \le 1$

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{it})\, h(e^{it})\,dt\right| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(u)\, g(e^{i(t-u)})\,du\right) h(e^{it})\,dt\right|$$

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$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(u) \underbrace{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(e^{i(t-u)})\, h(e^{it})|\,dt\right)}_{\le \|g_u\|_p \|h\|_q \le \|g\|_p} du \le \|g\|_p$$