I.2 Boundary behavior of holomorphic functions

Theorem 10 (on boundary behaviour of harmonic functions).

(1) *Let f be a bounded harmonic function on $U(0,1)$. Then there is a unique $f^* \in L^\infty(\mathbb{T})$ such that $P[f^*] = f$.*

(2) *Let $g \in L^1(\mathbb{T})$. Then $g(t) = \lim_{r\to 1-} P[g](re^{it})$ for almost all $t \in [0, 2\pi)$ (with respect to the Lesgue measure).*

Corollary (Fatou theorem). *Let f be a bounded holomorphic function on $U(0,1)$. Then the limit $f^*(e^{it}) = \lim_{r\to 1-} f(re^{it})$ exists for almost all $t \in [0, 2\pi)$. Moreover, $f^* \in L^\infty(\mathbb{T})$, $\|f^*\|_\infty = \sup\{|f(z)| : z \in U(0,1)\}$ and for each $z \in U(0,1)$ we have*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f^*(e^{it})}{e^{it}-z} e^{it}\,\mathrm{d}t.$$

Corollary. *Let f be a bounded holomorphic function on $U(0,1)$ and f^* be the function provided by the Fatou theorem. If $f^* = 0$ almost everywhere on some arc, then $f = 0$ on $U(0,1)$.*

Lemma 11. *Let $G \subset \mathbb{C}$ be a bounded simply connected domain, let f be a conformal mapping of G onto $U(0,1)$. Let $\varphi : [0,1] \to \mathbb{C}$ be a continuous curve satisfying $\varphi([0,1)) \subset G$ and $\varphi(1) \in \partial G$. Then the limit $\lim_{t\to 1-} f(\varphi(t))$ exists and its value belongs to the unit circle.*

Definition. Let $G \subset \mathbb{C}$ be an open set and $w \in \partial G$. We say that

(i) the point w is **accessible**, if there exists a continuous curve $\varphi : [0,1] \to \mathbb{C}$ such that $\varphi([0,1)) \subset G$ and $\varphi(1) = w$;

(ii) the point w is **simple**, if for any sequence of points $z_n \in G$ such that $z_n \to w$ there exists a continuous curve $\varphi : [0,1] \to \mathbb{C}$ satisfying $\varphi([0,1)) \subset G$ and $\varphi(1) = w$ and, moreover, there exists a strictly increasing sequence of points $t_n \in (0,1)$ such that $t_n \to 1$ and $\varphi(t_n) = z_n$ for each $n \in \mathbb{N}$.

Theorem 12. *Let $G \subset \mathbb{C}$ be a bounded simply connected domain, let f be a conformal mapping of G onto $U(0,1)$.*

(1) *If $w \in \partial G$ is simple, the mapping f can be continuously extended on $G \cup \{w\}$. After extending we have $|f(w)| = 1$.*

(2) *If $w_1, w_2 \in \partial G$ are two distinct simple point, the mapping f can be continuously extended on $G \cup \{w_1, w_2\}$. After extending we have $f(w_1) \neq f(w_2)$.*

Corollary. *Let $G \subset \mathbb{C}$ be a bounded simply connected domain such that any point $w \in \partial G$ is simple. Then any conformal mapping of G onto $U(0,1)$ can be extended to a homeomorphism of $\overline{G}$ onto $\overline{U(0,1)}$.*