Příklad E 1

(a) $T: \ell^\infty \to \ell^\infty$ $\quad Tx = (x_1 x_2, x_3 x_4, x_5 x_6, \dots)$

není lineární

$T(1,1,\dots) = (1,1,1,\dots)$

$T(2,2,\dots) = (4,4,4,\dots) \neq 2\cdot(1,1,1,\dots)$

(b) $T: L^1([0,\infty)) \to L^1([0,\infty))$ $\quad Tf(t) = e^{-t} \cdot f(t)$, $t \in [0,\infty)$, $f \in L^1$

$$\|Tf\|_1 = \int_0^\infty |e^{-t} f(t)|\,dt \leq \int_0^\infty |f(t)|\,dt = \|f\|_1$$

$\Rightarrow Tf \in L^1$, T je zjevně lineární, $\|T\| \leq 1$.

$\|T\| = 1$: zvolme $r > 0$ libovolné a $f := \frac{1}{r} \chi_{[0,r]}$

$\Rightarrow \|f\|_1 = 1$

$$\|Tf\| = \int_0^\infty e^{-t} |f(t)|\,dt = \int_0^r \frac{1}{r} e^{-x}\,dx \geq e^{-r}$$

$\Rightarrow \|T\| \geq e^{-r}$ | protože $r > 0$ je lib., je $\|T\| \geq 1$, tedy $\|T\| = 1$

Norma se nenabývá: Nechť $f \in L^1([0,\infty))$, $\|f\|_1 \leq 1$, $\|Tf\|_1 = 1$

Pro každé $r > 0$ je

$$1 = \|Tf\|_1 = \int_0^\infty e^{-t}|f(t)|\,dt = \int_0^r e^{-t}|f(t)|\,dt + \int_r^\infty e^{-t}|f(t)|\,dt$$

$$\leq \int_0^r |f(t)|\,dt + \int_r^\infty e^{-r}|f(t)|\,dt = \int_0^\infty |f(t)|\,dt - (1-e^{-r}) \int_r^\infty |f(t)|\,dt$$

$$\leq 1 - (1-e^{-r}) \int_r^\infty |f(t)|\,dt \leq 1$$

$\Rightarrow \int_r^\infty |f(t)|\,dt = 0 \Rightarrow f(t) = 0$ s.v. na (r,∞)

$r > 0$ je l.s. $\Rightarrow f = 0$ s.v. na $(0,\infty)$

$\left((0,\infty) = \bigcup_{n=1}^\infty (\frac{1}{n},\infty) \right)$

$\Rightarrow f = 0$ $\lightning$

(e) $T: c_0 \to \mathbb{C}$ $\quad Tx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x_n$, $(x_n) = x \in c_0$

Protože $\left(\frac{(-1)^n}{n}\right) \notin \ell^1$, není to spojitý lin. funkcionál.

T např. $(-1, 1, -1, \dots, (-1)^n, 0, \dots) \in B_{c_0}$

$$T(\dots) = 1 + \frac{1}{2} + \dots + \frac{1}{n} \to \infty$$

(d) $X = Y = L^2(\mathbb{R}, m)$, T je transformace z Plancherelovy věty

z věty plyne, že T je izometrie $\Rightarrow \|T\| = 1$,

nabývá se v každém bodě S_X

Příklad E2 $\quad X = c_0$, $Tx = (i x_1, \frac{1}{3} x_3, \frac{1}{4} x_4, \frac{1}{5} x_5, \dots)$

(a) T je spojitý lin. operátor z X do X:

T je do c_0: $\quad x \in c_0 \Rightarrow x_k \to 0 \Rightarrow \frac{x_k}{k} \to 0$

T je zřejmě lineární, $\quad \|Tx\|_\infty = \max\{|x_1|, \max\{|\frac{x_k}{k}| : k \geq 3\}\}$

$$\leq \|x\|_\infty$$

$$\Rightarrow \|T\| \leq 1$$

(b) T je kompaktní:

$$T^k x = (i x_1, \frac{1}{3} x_3, \dots, \frac{1}{k} x_k, 0, \dots), \quad k \geq 3$$

$\Rightarrow T^k$ je konečně-dimenzionální spojitý operátor, tedy kompaktní

$$\|T^k x - Tx\|_\infty = \|(0, \dots, 0, -\frac{1}{k+1} x_{k+1}, -\frac{1}{k+2} x_{k+2}, \dots)\|_\infty$$

$$\leq \frac{1}{k+1} \|x\|_\infty \Rightarrow \|T^k - T\| \leq \frac{1}{k+1} \Rightarrow T^k \to T \text{ v } L(X)$$

$\Rightarrow T$ je kompaktní

(c) $\sigma_p(T)$ in $\sigma(T)$

$\sigma_p(T)$: $Tx = \lambda x$:

$$
\begin{array}{l|l}
i x_1 = \lambda x_1 & \lambda = 0 \\
\frac{1}{3} x_3 = \lambda x_2 & \Rightarrow x_1 = 0 \\
\frac{1}{4} x_4 = \lambda x_3 & x_3 = x_4 = \dots = 0 \\
\vdots & (0,1,0,\dots) \text{ je lastni vektor} \\
\frac{1}{k+1} x_{k+1} = \lambda x_k & \Rightarrow 0 \in \sigma_p(T)
\end{array}
$$

$\lambda \neq 0$: $x_1 \neq 0 \Rightarrow \lambda = i$ ... $(1,0,0,\dots)$ je lastni vektor pri $\lambda = i$ $\Rightarrow i \in \sigma_p(T)$

$\lambda \neq 0, i$: $x_1 = 0$

$x_3 = 3\lambda x_2$

$x_4 = 4\lambda x_3 = 4\cdot 3 \cdot \lambda^2 x_2$

itd.

$x_{k+1} = (k+1)\cdot k \cdot \ldots \cdot 4 \cdot 3 \cdot \lambda^{k-2} \cdot x_2$

$$\Rightarrow x_{k+1} = \frac{(k+1)!}{2} \lambda^{k-2} \cdot x_2$$

če je $x_2 = 0$, potem $x = 0$

če $x_2 \neq 0$, potem $\left| \frac{(k+1)!}{2} \lambda^{k-2} x_2 \right| \to \infty$,

ker je $\left| \dfrac{\frac{(k+2)!}{2} \lambda^{k-1} x_2}{\frac{(k+1)!}{2} \lambda^{k-2} x_2} \right| = (k+2)|\lambda| \to \infty$

Zato: $\sigma_p(T) = \{0, i\}$

Ker je T kompakten, je $\sigma(T) = \{0\} \cup \sigma_p(T) \Rightarrow \sigma(T) = \{0, i\}$

Příklad E3 $\quad F$ Fourierova transf. na $L^1(\mathbb{R}, m_1)$

P Plancherelova transf. na $L^2(\mathbb{R}, m_1)$

$$f = \chi_{(0,1)} - \chi_{(-1,0)}$$

(b) $f \in L^1 \cap L^2 \Rightarrow Ff = Pf \in L^2$

(a) $$Pf(t) = Ff(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-itx} dx = \frac{1}{\sqrt{2\pi}} \left(\int_{-1}^{0} -e^{-itx} dx + \int_0^1 e^{-itx} dx \right) \overset{t=0}{=} \frac{1}{\sqrt{2\pi}}(-1+1) = 0$$

$$\overset{t\neq 0}{=} \frac{1}{\sqrt{2\pi}} \left(\left[\frac{e^{-itx}}{it} \right]_{-1}^{0} + \left[\frac{e^{-itx}}{-it} \right]_0^1 \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1 - e^{it}}{it} + \frac{e^{-it} - 1}{-it} \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{2 - e^{it} - e^{-it}}{it} \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2 - 2\cos t}{it} = \sqrt{\frac{2}{\pi}} \cdot \frac{1 - \cos t}{it}$$

(c) Dle Plancherelovy věty $\|Pf\|_2 = \|f\|_2$

$$\|Pf\|_2 = \left(\int_{\mathbb{R}} \frac{2}{\pi} \left| \frac{1 - \cos t}{t} \right|^2 dt \right)^{1/2} = \left(\frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_{\mathbb{R}} \left| \frac{1 - \cos t}{t} \right|^2 dt \right)^{1/2}$$

$$\|f\|_2 = \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f|^2 \right)^{1/2} = \left(\frac{1}{\sqrt{2\pi}} \cdot 2 \right)^{1/2} = \left(\sqrt{\frac{2}{\pi}} \right)^{1/2}$$

$$\Rightarrow \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_{\mathbb{R}} \left| \frac{1 - \cos t}{t} \right|^2 dt = \sqrt{\frac{2}{\pi}} \Rightarrow \int_{\mathbb{R}} \left| \frac{1 - \cos t}{t} \right|^2 dt = \pi$$

(d). Protože $f \in L^1$, je μ_f temperovaná

$$F\mu_f(\varphi) = \mu_f(F\varphi) = \int_{\mathbb{R}} f \cdot F\varphi = \int_{\mathbb{R}} Ff \cdot \varphi = \mu_{Ff}(\varphi)$$

$\Rightarrow F\mu_f = \mu_{Ff} = \mu_{Pf}$, Pf spočteno (a).