FUNCTIONAL ANALYSIS 2

SUMMER SEMESTER 2023/2024

PROBLEMS TO CHAPTER XIII

Problems to Section XIII.1 – measurable calculus for bounded normal operators

Problem 1. Let $H = \ell^2$ and let $\boldsymbol{z} = (z_n)$ be a bounded sequence of complex numbers. For $\boldsymbol{x} = (x_n) \in H$ set $M_{\boldsymbol{z}}(\boldsymbol{x}) = (z_n x_n)$. We know (from Problems 15, 25 and 31 to Chapter XII) that $M_{\boldsymbol{z}}$ is a bounded normal operator, $\|M_{\boldsymbol{z}}\| = \|\boldsymbol{z}\|_\infty$ and $\sigma(M_{\boldsymbol{z}}) = \overline{\{z_n; n \in \mathbb{N}\}}$.

(1) Let $f \in \mathcal{C}(\sigma(M_{\boldsymbol{z}}))$. Show that $\widetilde{f}(M_{\boldsymbol{z}}) = M_{f\circ\boldsymbol{z}}$, where $f \circ \boldsymbol{z} = (f(z_n))$.
(2) Let $(\boldsymbol{e}_n)$ be the canonical orthonormal basis of H. Show that $E_{\boldsymbol{e}_n,\boldsymbol{e}_n} = \delta_{z_n}$ (the Dirac measure supported by z_n) and $E_{\boldsymbol{e}_n,\boldsymbol{e}_m} = 0$ for $m, n \in \mathbb{N}$, $m \neq n$.
(3) Let $\boldsymbol{x} = (x_n) \in H$ and $\boldsymbol{y} = (y_n) \in H$. Show that $E_{\boldsymbol{x},\boldsymbol{y}} = \sum_{n=1}^\infty x_n \overline{y_n} \delta_{z_n}$.
(4) Deduce that $\mathcal{A}$, the domain σ-algebra of the spectral measure of $M_{\boldsymbol{z}}$ contains all subsets of $\sigma(M_{\boldsymbol{z}})$.
(5) Let $g : \sigma(M_{\boldsymbol{z}}) \to \mathbb{C}$ be any bounded function. Show that $\widetilde{g}(M_{\boldsymbol{z}}) = M_{g\circ\boldsymbol{z}}$.
(6) Let $A \subset \sigma(M_{\boldsymbol{z}})$ be arbitrary. Show that $E(A)$ (the value of the spectral measure of $M_{\boldsymbol{z}}$) is given by

$$E(A)(\boldsymbol{x}) = \sum_{n\in\mathbb{N}, z_n\in A} x_n \boldsymbol{e}_n, \quad \boldsymbol{x} = (x_n) \in H.$$

`Hint:` *(1) Use the 'moreover part' of Theorem XI.14. (3) For finitely supported vectors use (2) and Proposition XIII.2(a,b). For general* $\boldsymbol{x}, \boldsymbol{y}$ *set* $\boldsymbol{x}_n = \sum_{j=1}^n x_j \boldsymbol{e}_j$ *and* $\boldsymbol{y}_n = \sum_{j=1}^n y_j \boldsymbol{e}_j$, *show that* $\langle T\boldsymbol{x}_n, \boldsymbol{y}_n\rangle \to \langle T\boldsymbol{x}, \boldsymbol{y}\rangle$ *for any* $T \in L(H)$ *and apply it for* $\widetilde{f}(M_{\boldsymbol{z}})$. *(6) Apply (5) to the characteristic function of* A.

Problem 2. Let $H = \ell^2(\Gamma)$, where Γ is any set (possibly uncountable). Let $\varphi : \Gamma \to \mathbb{C}$ be a bounded function. For $f \in H$ set $M_\varphi(f) = \varphi \cdot f$.

(1) Show that this is a special case of the operator from Problem 17 to Chapter XII. Deduce that M_φ is a bounded linear operator and $\|M_\varphi\| = \|\varphi\|_\infty$.
(2) Using Problem 32 to Chapter XII show that M_φ is a normal operator.
(3) Using Problem 26 to Chapter XII show that $\sigma(M_\varphi) = \overline{\varphi(\Gamma)}$.
(4) Let $f \in \mathcal{C}(\sigma(M_\varphi))$. Show that $\widetilde{f}(M_\varphi) = M_{f\circ\varphi}$.
(5) Let $(\boldsymbol{e}_\gamma)_{\gamma\in\Gamma}$ be the canonical orthonormal basis of H. Show that $E_{\boldsymbol{e}_\gamma,\boldsymbol{e}_\gamma} = \delta_{\varphi(\gamma)}$ (the Dirac measure supported by $\varphi(\gamma)$) and $E_{\boldsymbol{e}_\gamma,\boldsymbol{e}_\delta} = 0$ for $\gamma, \delta \in \Gamma$, $\gamma \neq \delta$.
(6) Let $f, g \in H$. Show that $E_{f,g} = \sum_{\gamma\in\Gamma} f(\gamma)\overline{g(\gamma)}\delta_{\varphi(\gamma)}$.
(7) Deduce that $\mathcal{A}$, the domain σ-algebra of the spectral measure of M_φ contains all subsets of $\sigma(M_\varphi)$.
(8) Let $g : \sigma(M_\varphi) \to \mathbb{C}$ be any bounded function. Show that $\widetilde{g}(M_\varphi) = M_{g\circ\varphi}$.
(9) Let $A \subset \sigma(M_\varphi)$ be arbitrary. Show that $E(A)$ (the value of the spectral measure of M_φ) equals $M_{\chi_{\varphi^{-1}(A)}}$.

`Hint:` *(4) Use the 'moreover part' of Theorem XI.14. (6) For finitely supported vectors use (5) and Proposition XIII.2(a,b). For general g,h and a finite set $F\subset\Gamma$ denote $g_F=\chi_F\cdot g$ and $h_F=\chi_F\cdot h$, show that $\langle Tg_F,h_F\rangle\xrightarrow{F}\langle Tg,h\rangle$ (this is the limit of a net indexed by the up-directed set of finite subsets of Γ) for any $T\in L(H)$ and apply it for $\widetilde{f}(M_\varphi)$. (9) Apply (8) to the characteristic function of A.*

Problem 3. Let $H=L^2((0,1))$ and let $\varphi:(0,1)\to\mathbb{C}$ be a bounded Lebesgue measurable function. For $f\in H$ set $M_\varphi(f)=\varphi\cdot f$.

(1) Show that this is a special case of the operator from Problem 17 to Chapter XII. Deduce that M_φ is a bounded linear operator and $\|M_\varphi\|=\|\varphi\|_\infty$.
(2) Using Problem 32 to Chapter XII show that M_φ is a normal operator.
(3) Using Problem 26 to Chapter XII show that $\sigma(M_\varphi)$ is the essential range of φ.
(4) In case φ is continuous on $(0,1)$, deduce that $\sigma(M_\varphi)=\overline{\varphi((0,1))}$.
(5) Let $f\in\mathcal{C}(\sigma(M_\varphi))$. Show that $\widetilde{f}(M_\varphi)=M_{f\circ\varphi}$.
(6) Assume that (ψ_n) is a uniformly bounded sequence of Lebesgue measurable functions on $(0,1)$ converging almost everywhere to a function ψ. Show that $M_{\psi_n}(f)\to M_\psi f$ in H for each $f\in H$.
(7) Let $g\in L^\infty(E_{M_\varphi})$. Show that $\widetilde{g}(M_\varphi)=M_{g\circ\varphi}$.
(8) Let $A\subset\sigma(M_\varphi)$ be a Borel set. Show that $E(A)$ (the value of the spectral measure of M_φ) equals $M_{\chi_{\varphi^{-1}(A)}}$.

`Hint:` *(5) Use the 'moreover part' of Theorem IV.38. (6) Use Lebesgue dominated convergence theorem. (7) Let (f_n) be a sequence provided by Lemma XIII.3(b). Next combine (5,6) and Theorem XIII.4(b). (8) Apply (7) to the characteristic function of A.*

Problem 4. Let $H=\ell^2(\mathbb{Z})$ and let $S\in L(H)$ be defined by $S((x_n))=(x_{n-1})$. By Problem 5 to Chapter XII we know that S is a unitary (hence normal) operator and $\sigma(S)=\mathbb{T}=\{\lambda\in\mathbb{C};|\lambda|=1\}$.

(1) Let $K=L^2((0,2\pi),\mu)$, where μ is the normalized Lebesgue measure. For $n\in\mathbb{Z}$ set $\varphi_n(t)=e^{int}$, $t\in(0,2\pi)$. Show that $U:H\to K$ defined by $U((x_n))=\sum_n x_n\varphi_n$ (the series is considered in the Hilbert space K) is a unitary operator.
(2) Show that $S=U^*M_{\varphi_1}U$.
(3) Compute the values of continuous or measurable calculus applied to S and the spectral measure of S.

`Hint:` *(1) Use known facts from the theory of Fourier series. (3) Use (2) and Problem 3. For example, $\tilde{f}(S)=U^*\tilde{f}(M_{\varphi_1})U$ etc.*

Problems to Section XIII.2 – integral with respect to a spectral measure

Problem 5. Let Γ be any set and $\varphi : \Gamma \to \mathbb{C}$ any mapping. For $A \subset \mathbb{C}$ let $E(A)$ be the projection on $\ell^2(\Gamma)$ defined by $E(A)(f) = f \cdot \chi_{\varphi^{-1}(A)}$, $f \in \ell^2(\Gamma)$, i.e.,

$$E(A)(f)(\gamma) = \begin{cases} f(\gamma) & \text{if } \varphi(\gamma) \in A, \\ 0 & \text{otherwise,} \end{cases} \qquad f \in \ell^2(\Gamma).$$

(1) Show that E is an abstract spectral measure in $\ell^2(\Gamma)$ defined on the σ-algebra of all subsets of $\mathbb{C}$.
(2) Show that E is compactly supported if and only if φ is bounded.
(3) Describe the measures $E_{f,g}$, $f, g \in \ell^2(\Gamma)$.
(4) Let $\psi : \mathbb{C} \to \mathbb{C}$ be any function. Show that $\int \psi \, \mathrm{d}E = M_{\psi \circ \varphi}$ (using the notation from Problem 17 to Chapter XII).

Problem 6. Let (Ω, Σ, μ) be a complete measure space with μ semifinite and let $\varphi : \Omega \to \mathbb{C}$ be a Σ-measurable function. Set $\mathcal{A} = \{A \subset \mathbb{C}; \varphi^{-1}(A) \in \Sigma\}$. For $A \in \mathcal{A}$ let $E(A) = M_{\chi_{\varphi^{-1}(A)}}$ (using the notation from Problem 17 to Chapter XII).

(1) Show that E is an abstract spectral measure in $L^2(\mu)$ defined on the σ-algebra $\mathcal{A}$.
(2) Show that E is compactly supported if and only if φ is essentially bounded.
(3) Describe the measures $E_{f,g}$, $f, g \in L^2(\mu)$.
(4) Let $\psi : \mathbb{C} \to \mathbb{C}$ be any $\mathcal{A}$-measurable function. Show that $\int \psi \, \mathrm{d}E = M_{\psi \circ \varphi}$.

Problem 7. (1) Show that the inclusion in Theorem XIII.12(a) may be strict.
(2) Show that the inclusion in Theorem XIII.12(b) may be strict.

`Hint:` *(1) Take f unbounded and $g = -f$. (2) Take g to be strictly positive and unbounded and $f = \frac{1}{g}$.*

Problem 8. Let T be a nonzero compact normal operator on a Hilbert space. Show that T may be expressed in the form

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x, x_n \rangle x_n, \quad x \in H,$$

where $N \in \mathbb{N}$ or $N = +\infty$, $(x_n)_{n=1}^{N}$ is an orthonormal system and λ_n are nonzero complex numbers satisfying $\lambda_n \to 0$ if $N = +\infty$. (I.e., an analogue of the Hilbert Schmidt theorem – Theorem III.38 from Introduction to functional analysis – holds, just the coefficients λ_n need not be real.)

`Hint:` *Proceed similarly as in the selfadjoint case – use the description of the spectrum of a compact operator (Theorem III.35) and the fact that eigenvectors associated to different eigenvalues are orthogonal (Proposition XII.4(d)).*

Problem 9. Let H be an infinite-dimensional Hilbert space and let T be a nonzero compact normal operator on H. Consider the formula for T provided by Problem 8.

(1) Show that $\sigma(T) = \{0\} \cup \{\lambda_n; n \in \mathbb{N}, n \leq N\}$.

(2) Let $f : \sigma(T) \to \mathbb{C}$ be a bounded function. Show that f is always Borel measurable and it is continuous if and only if $f(\lambda_n) \to 0$.

(3) Let $f : \sigma(T) \to \mathbb{C}$ be a continuous function satisfying $f(0) = 0$. Show that $\widetilde{f}(T)$ is also a compact operator and that

$$\widetilde{f}(T)x = \sum_{n=1}^{N} f(\lambda_n) \langle x, x_n \rangle x_n, \quad x \in H.$$

(4) Let $f : \sigma(T) \to \mathbb{C}$ be a continuous function satisfying $f(0) \neq 0$. Show that $f(0)I - \widetilde{f}(T)$ is a compact operator (and hence $\widetilde{f}(T)$ is not compact) and that

$$\begin{aligned}\widetilde{f}(T)x &= f(0)Px + \sum_{n=1}^{N} f(\lambda_n) \langle x, x_n \rangle x_n \\ &= f(0)x + \sum_{n=1}^{N} (f(\lambda_n) - f(0)) \langle x, x_n \rangle x_n, \quad x \in H,\end{aligned}$$

where P is the orthogonal projection onto $\operatorname{Ker} T$.

`Hint:` *(3),(4): Use Theorems XI.14 and XI.15, in particular their 'moreover parts'.*

Problem 10. Let H and T be as in Problem 9.

(1) Let E be the spectral measure of T. Show that E is defined on the σ-algebra of all subsets of $\mathbb{C}$ and that, given $A \subset \mathbb{C}$, $E(A)$ is the orthogonal projection onto

$$\overline{\operatorname{span} \bigcup \{\operatorname{Ker}(\lambda_n I - T); \lambda_n \in A\}}$$

if $0 \notin A$ and the orthogonal projection onto

$$\operatorname{Ker} T \oplus \overline{\operatorname{span} \bigcup \{\operatorname{Ker}(\lambda_n I - T); \lambda_n \in A\}}$$

if $0 \in A$.

(2) Let $f : \sigma(T) \to \mathbb{C}$ be a bounded function. Show that $\widetilde{f}(T)$ may be expressed by the formula from Problem 9(4).

(3) Characterize bounded functions f for which $\widetilde{f}(T)$ is a compact operator.

(4) Let $f : \sigma(T) \to \mathbb{C}$ be an unbounded function. Describe the operator $\int f \, \mathrm{d}E$ (i.e., find its domain and a formula).

Problems to Section XIII.4 - unbounded normal operators

Problem 11. Consider the operators $T = M_{\boldsymbol{z}}$ from Problem 15 to Chapter XII on ℓ^2 or $T = M_g$ from Problem 17 to Chapter XII on $L^2(\mu)$.

(1) Show that these operators are normal.

(2) Compute the operators B and C from Lemma XIII.19.

(3) Compute the projections P_j from Theorem XIII.21.

Problem 12. Let E be an abstract spectral measure in a Hilbert space H defined on a σ-algebra $\mathcal{A}$. Let $f : \mathbb{C} \to \mathbb{C}$ be an $\mathcal{A}$-measurable function and $T = \int f \, \mathrm{d}E$.

(1) Show that the operator T is normal.
(2) Compute the operators B and C from Lemma XIII.19.
(3) Compute the projections P_j from Theorem XIII.21.

Problem 13. Let $k \in \mathbb{Z}$ and let $\boldsymbol{z} = (z_n)_{n\in\mathbb{Z}}$ be a fixed sequence of complex numbers. Define an operator T on $\ell^2(\mathbb{Z})$ by the formula

$$T((x_n)_{n\in\mathbb{Z}}) = (z_{n+k}x_{n+k})_{n\in\mathbb{Z}}, \quad (x_n) \in D(T) = \{(y_n) \in \ell^2(\mathbb{Z}); (z_{n+k}y_{n+k}) \in \ell^2(\mathbb{Z}).$$

(1) Show that T is a closed densely defined operator.
(2) Compute T^*.
(3) Compute T^*T and TT^*.
(4) Under which conditions is T normal? Under which conditions is T self-adjoint?
(5) Compute the operators B and C from Lemma XIII.19.

Problem 14. Let $r \in \mathbb{R}$ and let $\psi : \mathbb{R} \to \mathbb{C}$ be a fixed measurable function. Define an operator T on $L^2(\mathbb{R})$ by the formula

$$T(f)(t) = \psi(t+r) \cdot f(t+r), \quad t \in \mathbb{R}, f \in L^2(\mathbb{R}).$$

(1) Show that T is a closed densely defined operator.
(2) Compute T^*.
(3) Compute T^*T and TT^*.
(4) Under which conditions is T normal? Under which conditions is T self-adjoint?
(5) Compute the operators B and C from Lemma VI.19.

Problems to Section XIII.5 - diagonalization of operators

Problem 15. Let H, K be two Hilbert spaces and let $U : H \to K$ be a unitary operator (i.e., an onto isometry). Let T be an operator on H. Set $S = UTU^{-1}$. Show that operator T and S have same properties. In particular:

(1) S is closed or densely defined if and only if T has the respective property.
(2) $\sigma(S) = \sigma(T)$ and S, T have the same eigenvalues.
(3) $S^* = UT^*U^{-1}$.
(4) S is selfadjoint (symmetric, maximal symmetric, normal) if and only if T has the respective property.
(5) Suppose that T is normal. Let E_T be the spectral measure of T, let $\mathcal{A}_T$ be its domain σ-algebra. Similary, let E_S be the spectral measure of S and $\mathcal{A}_S$ be its domain σ-algebra. Then $\mathcal{A}_S = \mathcal{A}_T$ and $E_S(A) = UE_T(A)U^{-1}$ for $A \in \mathcal{A}_T$.

Problem 16. Let $U : L^2((0, 2\pi)) \to \ell^2(\mathbb{Z})$ be the isometry known from the theory of Fourier series, i.e.,

$$U(f)(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-int} \, \mathrm{d}t, \quad n \in \mathbb{Z}, f \in L^2((0, 2\pi)).$$

Consider the operator T_j, $j = 1, \dots, 6$, on $L^2((0, 2\pi))$ defined analogously as the operators from Problem 22 to Chapter XII.

(1) Compute the operators UT_jU^{-1} for $j = 1, \dots, 6$.
(2) Compute the spectral measure of the operator T_5.

`Hint:` *(1) Use integration by parts. Moreover, for $f \in D(T_1)$ show that $f(0) - f(2\pi) = \lim_{n\to\pm\infty} 2\pi i n \hat{f}(n)$ and that $\frac{1}{2}(f(0) + f(2\pi)) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$ (by $\hat{f}(n)$ we denote the Fourier coefficients; to prove the second equality use the Jordan-Dirichlet criterion).*

Problem 17. Let $\mathcal{P} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Plancherel transform, i.e., the extension to $L^2(\mathbb{R})$ of the Fourier transform restricted to $L^1(\mathbb{R}) \cap L^2(R)$. It can be expressed by the formula

$$\mathcal{P}(f) = \lim_{r\to\infty} \left(t \mapsto \frac{1}{2\pi} \int_{-r}^{r} f(x) e^{-itx} \,\mathrm{d}x \right) \quad \text{(the limit taken in } L^2(\mathbb{R})\text{)}.$$

Consider the operator T_1 from Problem 24 to Chapter XII on $L^2(\mathbb{R})$.

(1) Compute the operators $\mathcal{P} T_1 \mathcal{P}^{-1}$.
(2) Compute the spectral measure of the operator T_1.