FUNCTIONAL ANALYSIS 2

SUMMER SEMESTER 2023/2024

PROBLEMS TO CHAPTER XI

Problem 1. Show that the Banach algebra $\ell^1(\mathbb{Z})$ (see Problem 12 to Chapter X) admits no involution and equivalent norm making it a C^*-algebra.

`Hint:` *Consider the Gelfand transform (see Problem 41 to Chapter X) and show that its range is a proper dense subspace of $\mathcal{C}(\mathbb{T})$. Then use Theorem XI.9.*

Problem 2. Show that the Banach algebra $L^1(\mathbb{R})$ (see Problem 14 to Chapter X) admits no involution and equivalent norm making it a C^*-algebra.

`Hint:` *Consider the Gelfand transform and its relationship to the Fourier transform (see Problem 43 to Chapter X), use the known fact, that the Fourier transform is not onto $\mathcal{C}_0(\mathbb{R}^n)$ and Theorem XI.9.*

Problem 3. Let $A = \mathcal{C}(K)$, let $g \in A$ and let F be a fuction continuous on $\sigma(g) = g(K)$. Show that $\widetilde{F}(g) = F \circ g$.

Problem 4.
1. Let A be a commutative C^*-algebra and $x \in A$. Show that x is self-adjoint if and only if $\sigma(x) \subset \mathbb{R}$.
2. Is this equivalence valid also for non-commutative C^*-algebras?

`Hint:` *(1) For $\Rightarrow$ use Proposition XI.8. For the converse note that the statement is valid in $\mathcal{C}_0(T)$ and use Theorem XI.9. (2) Find a counterxample to $\Leftarrow$ in the matrix algebra M_2.*

Problem 5. Let A be a C^*-algebra. An element $x \in A$ is called **positive** if it is self-adjoint and $\sigma(x) \subset [0, +\infty)$.

Show that each self-adjoint element may be expressed as a difference of two positive elements.

`Hint:` *Use the continuous function calculus for functions $t \mapsto t^+$ and $t \mapsto t^-$.*

Problem 6. Let A be a unital C^*-algebra, $x \in A$ a normal element and $f \in \mathcal{C}(\sigma(x))$.
1. Show that $\widetilde{f}(x)$ is a normal element.
2. Show that $\widetilde{f}(x)$ is a self-adjoint element if and only if the function f attains only real values.
3. Show that $\widetilde{f}(x)$ is a positive element if and only if the function f attains only non-negative values.
4. Show that $\widetilde{f}(x)$ is an invertible element if and only if the function f does not attain the value 0.

Problem 7. Let A be a C^*-algebra and let $x, y \in A$ be two positive elements which commute (i.e., $xy = yx$). Show that xy is a positive element as well.

`Hint:` *Let B be a closed subalgebra of A generated by x and y. Then B is a commutative C^*-algebra. Use Theorem XI.9 and the fact that the statement is valid in $\mathcal{C}_0(\Omega)$.*

Problem 8. Let A be a C^*-algebra with a unit e and let $x \in A$ be a normal element. Denote by B the closed subalgebra of A generates by the elements x, x^*, e and by B_0 the closed subalgebra of A generates by the elements x, x^*.

(1) Show that $B = B_0$ if and only if x is invertible in A.
(2) Let $f \in \mathcal{C}(\sigma(x))$. Show that $\tilde{f}(x) \in B$.
(3) Let $f \in \mathcal{C}(\sigma(x))$. Show that $\tilde{f}(x) \in B_0$ if and only if either x is invertible or $f(0) = 0$.

`Hint:` *(1) The implication $\Leftarrow$ is obvious. If x is not invertible, deduce from Theorem XI.15 that B_0 does not contain e. (2) Use Theorem XI.14. (3) The implication $\Leftarrow$ follows from (1) and Theorem XI.15. Assume that x is not invertible (i.e., $0 \in \sigma(x)$) and $f(0) \neq 0$. Since the function calculus is an isomorphism of $\mathcal{C}(\sigma(x))$ onto B and simultaneously an isomorphism of $\mathcal{C}_0(\sigma(x) \setminus \{0\})$ onto B_0, it is enough to observe that $f \notin \mathcal{C}_0(\sigma(x) \setminus \{0\})$.*

Problem 9. Consider the situation from Problem 8 and, moreover, fix $f \in C(\sigma(x))$. Denote by D the closed subalgebra of A generates by the elements $\tilde{f}(x), \tilde{f}(x)^*, e$ and by D_0 the closed subalgebra of A generates by the elements $\tilde{f}(x), \tilde{f}(x)^*$.

(1) Show that $D \subset B$ and $D_0 \subset B_0$.
(2) Show that $D = D_0$ if and only if f does not attain the value zero.
(3) Consider the diagram

$$\begin{array}{ccc} \Delta(B) & \xrightarrow{h_x} & \sigma(x) \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta(D) & \xrightarrow[h_{\tilde{f}(x)}]{} & \sigma(\tilde{f}(x)) \end{array} ,$$

where $r(\varphi) = \varphi|_D$ for $\varphi \in \Delta(B)$ and h_x and $h_{\tilde{f}(x)}$ are the mapping from the construction of the continuous calculus in Theorem XI.14, i.e., $h_x(\varphi) = \varphi(x)$ and $h_{\tilde{f}(x)}(\psi) = \psi(\tilde{f}(x))$. Show that this diagram commutes, i.e., $f \circ h_x = h_{\tilde{f}(x)} \circ r$.
(4) Deduce that for each $g \in \mathcal{C}(\sigma(\tilde{f}(x)))$ we have $\tilde{g}(\tilde{f}(x)) = \widetilde{g \circ f}(x)$.

`Hint:` *(2) Use Problem 8(1). (3) It is necessary to show that $\varphi(\tilde{f}(x)) = f(\varphi(x))$. To this end use the definition of the continuous function calculus and the definition of the Gelfand transform. (4) Use (3) and the definition of the continuous function calculus.*

Problem 10. Let A be a C^*-algebra and let $x \in A$ be a positive element.

(1) Show that there exists a positive element $y \in A$ such that $y^2 = x$.
(2) Is such y unique?

`Hint:` *(1) Use the continuous calculus for the function $t \mapsto \sqrt{t}$ to the element x. (2) Let z be a positive element satisfying $z^2 = x$ and let y be the element obtain by the way described in (1). Using Problem 9(4) applied to $f(t) = t^2$ and $g(t) = \sqrt{t}$ show that $z = y$.*