X.2 Spectrum and its properties

Definition. Let A be a Banach algebra with a unit e and let $x \in A$.

- The **spectrum** of x is the set
  $(\ \sigma_A(x) =\ )\quad \sigma(x) = \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not invertible in } A\}.$
- The **resolvent set** of x is the set $\rho(x) = \mathbb{C} \setminus \sigma(x)$.
- The **resolvent function** of x is the function
  $$R(\lambda, x) := (\lambda e - x)^{-1}, \qquad \lambda \in \rho(x).$$

If A is a Banach algebra without a unit and $x \in A$, the **spectrum** of x is defined by

$$(\ \sigma_A(x) =\ )\quad \sigma(x) = \sigma_{A^+}((x, 0)).$$

If A is a Banach algebra (unital or not), the **spectral radius** of $x \in A$ is the number

$$r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

Remarks:

(1) The spectrum of x is a purely algebraic notion, it does not depend on the norm of the Banach algebra.
(2) If A has no unit, then $0 \in \sigma_A(x)$ for each $x \in A$.
(3) If A is unital, then $\sigma_{A^+}((x, 0)) = \sigma_A(x) \cup \{0\}$ for $x \in A$.

Proposition 8 (properties of the resolvent function). *Let A be a unital Banach algebra and let $a \in A$. Then:*

(i) *$\rho(a)$ is an open subset of $\mathbb{C}$.*
(ii) *The mapping $\lambda \mapsto R(\lambda, a)$ is continuous on $\rho(a)$.*
(iii) *For $\lambda, \mu \in \rho(a)$ one has*

$$R(\mu, a) - R(\lambda, a) = -(\mu - \lambda) R(\mu, a) R(\lambda, a).$$

In particular, $R(\mu, a)$ and $R(\lambda, a)$ commute.
(iv) *The function $\lambda \mapsto \varphi(R(\lambda, a))$ is holomorphic on $\rho(a)$ for each $\varphi \in A^*$.*
(v) *For $|\lambda| > \|a\|$ one has $\lambda \in \rho(a)$ and*

$$R(\lambda, a) = \frac{1}{\lambda}\left(e - \frac{a}{\lambda}\right)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}.$$

(vi) *$aR(\lambda, a) = R(\lambda, a)a$ for any $\lambda \in \rho(a)$.*

Theorem 9. *Let A be a Banach algebra. Then for each $x \in A$ the spectrum $\sigma(x)$ is a nonempty compact subset of $\mathbb{C}$.*

Remark. $\{a \in A : \sigma(a) \subset G\}$ is open whenever $G \subset \mathbb{C}$ is open (i.e., „$\sigma : A \to \mathcal{K}(\mathbb{C})$ is an upper semi-continuous mutlivalued mapping of A to the set of all the nonempty compact subsets of $\mathbb{C}$").

Theorem 10 (Gelfand-Mazur). *Let A be a unital Banach algebra. Then A is a field (i.e., all the nonzero elements of A are invertible, in other words $G(A) = A \setminus \{0\}$), if and only if A is isometrically isomorphic to the Banach algebra $\mathbb{C}$.*

Definition. Let A be a Banach algebra, let $p(\lambda) = \sum_{j=0}^{n} \alpha_j \lambda^j$ be a polynomial with complex coefficients and let $a \in A$.

- If A has a unit e, we define $p(a) = \sum_{j=0}^{n} \alpha_j a^j$ (where $a^0 = e$).
- If A has no unit and $p(0) = 0$ (i.e., $\alpha_0 = 0$), we define $p(a) = \sum_{j=1}^{n} \alpha_j a^j$.

Lemma 11. *Let A be a unital Banach algebra and let p, q be polynomials with complex coefficients. Then $(pq)(a) = p(a)q(a)$ for each $a \in A$.*

Lemma 12 (spectrum and polynomials). *Let A be a unital Banach algebra, let p be a polynomial with complex coefficients and let $a \in A$. Then:*

(a) *$p(a) \in G(A)$, if and only if the roots of p belong to $\rho(a)$.*
(b) *$\sigma(p(a)) = p(\sigma(a))$.*

Theorem 13 (on the spectral radius). *Let A be a Banach algebra and let $a \in A$. Then:*

(a) $r(a) = \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}} = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}}$.
(b) *If A is unital, the formula from Proposition 8(v) holds also for $|\lambda| > r(a)$, where the series on the right-hand side converges absolutely.*

Corollary 14. *If A is a unital Banach algebra and an element $a \in A$ satisfies $r(a) < 1$, then $(e - a)^{-1} = \sum_{n=0}^{\infty} a^n$ (the series converges absolutely).*

Proposition 15. *Let A be Banach algebra with a unit e. Let B be a closed subalgebra of A containing e and let $x \in B$. Then:*

(a) $\partial \sigma_B(x) \subset \sigma_A(x) \subset \sigma_B(x)$.
(b) *Let G be a connected component of $\mathbb{C} \setminus \sigma_A(x)$. Then either $G \subset \sigma_B(x)$ or $G \cap \sigma_B(x) = \emptyset$.*
(c) *If $\mathbb{C} \setminus \sigma_A(x)$ is a connected set, then $\sigma_A(x) = \sigma_B(x)$.*

Corollary 16. *Let A be a Banach algebra, let B be its closed subalgebra and let $x \in B$. Then the assertions (a)–(c) of Proposition 14 hold, if we replace $\sigma_A(x)$ and $\sigma_B(x)$ by $\sigma_A(x) \cup \{0\}$ and $\sigma_B(x) \cup \{0\}$.*