XIII.3 Spectral decomposition of an (unbounded) selfadjoint operator

Proposition 14 (measurable calculus via integral). *Let $T \in L(H)$ be a normal operator. Let E_T be its spectral measure and $\mathcal{A}_T$ the respective σ-algebra. If g is a bounded $\mathcal{A}_T$-measurable function, then $\tilde{g}(T) = \int g \, \mathrm{d}E_T$.*

Lemma 15. *Let T be a selfadjoint operator on H. Let E be the spectral measure of the operator C_T. Then*

$$T = \int i \frac{1+z}{1-z} \, \mathrm{d}E(z).$$

Lemma 16 (on the image of a spectral measure). *Let F be an abstract spectral measure in H defined on a σ-algebra $\mathcal{A}$ and let $\varphi : \mathbb{C} \to \mathbb{C}$ be an $\mathcal{A}$-measurable mapping. Define*

$$\mathcal{A}' = \{A \subset \mathbb{C} : \varphi^{-1}(A) \in \mathcal{A}\}$$

and for $A \in \mathcal{A}'$ set

$$E(A) = F(\varphi^{-1}(A)).$$

Then E is an abstract spectral measure in H and for each $\mathcal{A}'$-measurable function f one has

$$\int f \, \mathrm{d}E = \int f \circ \varphi \, \mathrm{d}F.$$

Theorem 17 (spectral decomposition of a selfadjoint operator). *If T is a selfadjoint operator on a Hilbert space H, then there exists a unique abstract spectral measure E in H such that $T = \int \mathrm{id} \, \mathrm{d}E$.*

This measure E is the image of the spectral measure of the operator C_T under the Borel mapping $z \mapsto i\frac{1+z}{1-z}$.

Corollary 18. *Let T be a selfadjoint operator on H. Then T is bounded if and only if $\sigma(T)$ is a bounded set.*