XII.3 Spectrum of an unbounded operator

Convention. In this section we consider only Banach spaces over $\mathbb{C}$.

Definition.

- Let X be a Banach space. By an **operator on** X we mean an operator from X to X.
- Let T be an operator on X.
  - By the **resolvent set** of the operator T we mean the set of all $\lambda \in \mathbb{C}$, for which the operator $\lambda I - T$ is one-to-one, onto and $(\lambda I - T)^{-1} \in L(X)$. It is denoted by $\rho(T)$.
  - By the **resolvent function** of the operator T we mean the mapping $$\lambda \mapsto R(\lambda, T) = (\lambda I - T)^{-1}, \qquad \lambda \in \rho(T).$$
  - By the **spectrum** of the operator T we mean the set $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

Remarks.

(1) If T is not closed, then $\rho(T) = \emptyset$ and $\sigma(T) = \mathbb{C}$.

(2) The resolvent set is sometimes defined in a different way. Sometimes it is required

(a) just that the operator $\lambda I - T$ is one-to-one and onto;

sometimes it is required

(b) that the operator $\lambda I - A$ is one-to-one, its range is dense and the inverse operator is continuous.

If T closed, then all three definitions coincide; for non-closed operators they give different notions. If the operator T is not closed, but has a closed extension, then its resolvent set according to (b) equals the resolvent set of $\overline{T}$; the resolvent set according to (a) is disjoint with the resolvent set of $\overline{T}$.

Proposition 14 (properties of resolvent function, resolvent set and spectrum). *Let T be an operator on X.*

(a) *Let $\mu \in \rho(T)$. Then for for $\lambda \in \mathbb{C}$, $|\lambda - \mu| < \frac{1}{\|(\mu I - T)^{-1}\|}$ one has $\lambda \in \rho(T)$ and*
$$(\lambda I - T)^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda - \mu)^n ((\mu I - T)^{-1})^{n+1}.$$

(b) *$\rho(T)$ is an open subset of $\mathbb{C}$ and $\sigma(T)$ is a closed subset of $\mathbb{C}$.*

(c) *The resolvent function $\lambda \mapsto (\lambda I - T)^{-1}$ is continuous on $\rho(T)$.*

(d) *For any $f \in X^*$ and $x \in X$ the function $\lambda \mapsto f((\lambda I - T)^{-1} x)$ is holomorphic on $\rho(T)$.*

Lemma 15 (empty spectrum and T^{-1}). *If T is a closed operator on X such that $\sigma(T) = \emptyset$, then $T^{-1} \in L(X)$ and $\sigma(T^{-1}) = \{0\}$.*