Differential operators on $L^2(0,1)$, $L^2(0,\infty)$, $L^2(\mathbb{R})$

$\boxed{A}$ Let $T_j(f) = f'$ be operators on $L^2(0,1)$ with domains

$$D(T_1) = AC[0,1]$$
$$D(T_2) = \{ f \in D(T_1);\ f(0) = 0 \}$$
$$D(T_3) = \{ f \in D(T_1);\ f(1) = 0 \}$$
$$D(T_4) = \{ f \in D(T_1);\ f(0) = f(1) = 0 \}$$
$$D(T_5) = \{ f \in D(T_1);\ f(0) = f(1) \}$$

(1)
$$\begin{tikzcd} & T_3 \arrow[dr, phantom, "\subset"] & \\ T_4 \arrow[ur, phantom, "\subset"] \arrow[r, phantom, "\subset"] \arrow[dr, phantom, "\subset"] & T_2 \arrow[r, phantom, "\subset"] & T_1 \\ & T_5 \arrow[ur, phantom, "\subset"] & \end{tikzcd}$$

. All of them are densely defined, as $D(T_4)$ contains $\mathcal{D}((0,1))$, which is dense in $L^2((0,1))$ by Lemma VII.1.

(2) $T_1^* \supset -T_4$, $T_2^* \supset -T_3$, $T_3^* \supset -T_2$, $T_4^* \supset -T_1$, $T_5^* \supset -T_5$

Let j,k be one of the suggested pairs of indices $(1,4), (2,3), (3,2), (4,1), (5,5)$, $f \in D(T_j)$, $g \in D(T_k)$. Then

$$\langle T_j f, g \rangle = \int_0^1 f' \bar{g} \underset{\substack{\text{integration by parts}\\ \text{for absolutely cts functions}}}{=} [f \bar{g}]_0^1 - \int_0^1 f \bar{g}' =$$

$$= \underbrace{f(1)\overline{g(1)} - f(0)\overline{g(0)}}_{=0 \text{ due to the choice of } j,k} - \int_0^1 f \bar{g}' = \langle f, -T_k g \rangle$$

Thus $g \in D(T_j^*)$ and $T_j^* g = -T_k g$.
So $T_j^* \supset -T_k$

(3) $\forall j:\ T_j^* \subset -T_1$

Let $g \in D(T_j^*)$. Then there is $h \in L^2((0,1))$ s.t.

$$\forall f \in D(T_j):\ \langle T_j f, g \rangle = \langle f, h \rangle$$

Define $H(t) = \int_0^t h$, $t \in [0,1]$. Then $H \in AC[0,1]$, $H' = h$ a.e.

And we have for each $f \in D(T_j)$: $\int_0^1 f' \bar{g} = \int_0^1 f \bar{h} = [f \bar{H}]_0^1 - \int_0^1 f' \bar{H} =$

integration by parts for AC functions

$$= f(1)\overline{H(1)} - f(0)\underbrace{\overline{H(0)}}_{=0} - \int_0^1 f'\bar{H} = f(1)\overline{H(1)} - \int_0^1 f'\bar{H}$$

So :

$$(*) \quad \forall f \in D(T_j) : \int_0^1 f'\bar{g} = f(1)\overline{H(1)} - \int_0^1 f'\bar{H}$$

Since $\mathcal{D}((0,1)) \subset D(T_j)$, we deduce

$$\forall \varphi \in \mathcal{D}((0,1)) : \int_0^1 \varphi' \bar{g} = -\int_0^1 \varphi' \bar{H}, \text{ i.e.}$$

$$\int_0^1 (\bar{g} + \bar{H})\varphi' = 0.$$

So, $\overline{g+H}$ has derivative zero in the sense of distributions so $\exists C \in \mathbb{F}$: $g + H = C$ a.e. (by Proposition VII.3)

Hence $g = C - H$, so $g \in AC[0,1] = D(T_1)$ and $g' = -H' = -h$ a.e., hence in $L^2(0,1)$.

So $T_j^* g = h = -g' = -T_1 g$.

4 by 2 and 3 we deduce $T_4^* = -T_1$

5 We know $g + H = C$ and $H(0) = 0$, so $C = g(0)$. If we plug $H = g(0) - g$ to $(*)$ we obtain

$$\forall f \in D(T_j) : \int_0^1 f'\bar{g} = f(1)(\overline{g(0)} - \overline{g(1)}) - \int_0^1 f'(\overline{g(0)} - \bar{g})$$

$$= f(1)(\overline{g(0)} - \overline{g(1)}) - \underbrace{\int_0^1 f'\overline{g(0)}}_{(f(1)-f(0))\overline{g(0)}} + \int_0^1 f'\bar{g} \quad \text{cancels out}$$

$$= -f(1)\overline{g(1)} + f(0)\overline{g(0)}$$

So, we get $\forall f \in D(T_j) : -f(1)\overline{g(1)} + f(0)\overline{g(0)} = 0$

(6) $T_2^* = -T_3$: $\supset$ by (2)

$\subset$: $g \in D(T_2^*) \Rightarrow g \in D(T_1)$ by (3), moreover by (5) we know

$\forall f \in D(T_2) : -f(1)\overline{g(1)} = 0$

Since there is $f \in D(T_2)$ with $f(1) \neq 0$, necessarily $g(1) = 0$

so $g \in D(T_3)$. By (3) we know $T_2^* g = -g'$,

So $T_2^* = -T_3$

(7) $T_3^* = -T_2$ is completely analogous

(8) $T_1^* = -T_4$: $\supset$ by (2)

$\subset$: $g \in D(T_1^*) \overset{(3)}{\Longrightarrow} g \in D(T_1)$. Moreover, by (5):

$\forall f \in D(T_1) : -f(1)\overline{g(1)} + f(0)\overline{g(0)} = 0$.

$\exists f_1 \in D(T_1) : f_1(1) = 0,\ f_1(0) \neq 0 \Rightarrow g(0) = 0$

$\exists f_2 \in D(T_2) : f_2(1) \neq 0,\ f_2(0) = 0 \Rightarrow g(1) = 0$

So $g \in D(T_4)$.

(9) $T_5^* = -T_5$: $\supset$ by (2)

$\subset$ $g \in D(T_5^*) \overset{(3)}{\Rightarrow} g \in D(T_1)$. Moreover, by (5):

$\forall f \in D(T_5) : -f(0)\left(\overline{g(1)} - \overline{g(0)}\right) = 0$

Since $\exists f \in D(T_5) : f(0) \neq 0$, we get $g(1) = g(0)$

so $g \in D(T_5)$

(10) $T_1, \dots, T_5$ are closed by Prop. XII.21(a) as all of them are adjoint operators.

(11) iT_4 is symmetric, not self-adjoint; iT_5 is self-adjoint.

(12) Eigenvalues: $T_j f = \lambda f$ means $f' = \lambda f$. Only solutions are $t \mapsto c \cdot e^{\lambda t}$, where $c \in \mathbb{C}$

So: • $\sigma_p(T_1) = \mathbb{C}$ ... for each λ the function $t \mapsto e^{\lambda t}$ belongs to $D(T_1)$

• $\sigma_p(T_2) = \sigma_p(T_3) = \sigma_p(T_4) = \emptyset$

as the boundary conditions yield constant zero function.

($f(t) = c\, e^{\lambda t} \Rightarrow f(0) = c$, $f(1) = c e^{\lambda}$. If $f(0) = 0$ or $f(1) = 0$, then $c = 0$, hence $f \equiv 0$)

• $\sigma_p(T_5)$: $f(0) = f(1) \Rightarrow c = c e^{\lambda}$, i.e. $e^{\lambda} = 1$

$\lambda \in \{2k\pi i,\ k \in \mathbb{Z}\}$

So $\sigma_p(T_5) = \{2k\pi i,\ k \in \mathbb{Z}\}$

(13) Spectrum: $\sigma(T_1) = \mathbb{C}$ (as $\sigma(T_1) \supset \sigma_p(T_1)$)

$\sigma(T_4) = \mathbb{C}$ ($\sigma(T_4) = \sigma(-T_1^*) = \{-\overline{\lambda},\ \lambda \in \sigma(T_1)\} = \mathbb{C}$)

$\sigma(T_2) = \emptyset$: $g \in L^2(0,1) \Rightarrow \exists!\, f \in AC[0,1]: f' - \lambda f = g$

[existence and uniqueness theorem, or elementary calculations using integration factor]

$\sigma(T_5)$: iT_5 self-adjoint $\Rightarrow \sigma(T_5) \subset i\mathbb{R}$ by Thm XII.25

$\lambda \in \mathbb{R}$, consider $i\lambda$. $g \in L^2(0,1)$

$$f' - i\lambda f = g$$

$$f'(t) e^{-i\lambda t} - i\lambda e^{-i\lambda t} f(t) = g(t) e^{-i\lambda t}$$

$$(f(t) e^{-i\lambda t})' = g(t) e^{-i\lambda t}$$

$$f(t) e^{-i\lambda t} - f(0) = \int_0^t g(s) e^{-i\lambda s}\, ds$$

$$f(t) = f(0) e^{i\lambda t} + e^{i\lambda t} \int_0^t g(s) e^{-i\lambda s}\, ds$$

$f(1) = f(0) \Rightarrow$

$$f(0) = f(0) e^{i\lambda} + e^{i\lambda} \int_0^1 g(s) e^{-i\lambda s}\, ds$$

$$f(0)(1 - e^{i\lambda}) = e^{i\lambda} \int_0^1 g(s) e^{-i\lambda s}\, ds$$

if $i\lambda \notin \sigma_p(T_5)$, then $1 - e^{i\lambda} \neq 0$,

so $$f(0) = \frac{e^{i\lambda}}{1 - e^{i\lambda}} \int_0^1 g(s) e^{-i\lambda s}\, ds$$

<u>Conclusion:</u> $\sigma(T_5) = \sigma_p(T_5)$.

B Let T_j be operators on $L^2(0,\infty)$ defined by
$T_j(f) = f'$, where
$D(T_1) = \{ f \in AC_{loc}[0,\infty) ;\ f, f' \in L^2(0,\infty)\}$
$D(T_2) = \{ f \in D(T_1) ;\ f(0) = 0 \}$

(1) $T_2 \subset T_1$, T_1, T_2 are densely defined, because $\mathcal{D}((0,\infty)) \subset D(T_2) \subset D(T_1)$ and $\mathcal{D}((0,\infty))$ is dense in $L^2(0,\infty)$ by Lemma VII.1.

(2) $f \in D(T_1) \Rightarrow \lim_{t \to +\infty} f(t) = 0$

$f \in D(T_1) \Rightarrow f$ cts on $[0,\infty)$; $f \in L^2(0,\infty) \Rightarrow |f|^2 \in L^1(0,\infty)$

$$\underset{\substack{\uparrow \\ |f|^2 \in AC[0,r]}}{|f(r)|^2 = |f(0)|^2} + \int_0^r (|f|^2)' \underset{\substack{\uparrow \\ |f|^2 = f\cdot\bar f}}{=} |f(0)|^2 + \int_0^r \underbrace{f'\cdot\bar f + f\cdot \bar f'}_{\in L^1(0,\infty) \text{ by Hölder}} \xrightarrow{r \to +\infty} |f(0)|^2 + \int_0^\infty f'\bar f + f\bar f'$$

So, $\lim_{r\to\infty} |f(r)|^2$ exists. Since $|f|^2 \in L^1 \Rightarrow$ the limit must be zero.

So $f(r) \to 0$ for $r \to \infty$

(3) $T_1^* \supset -T_2$, $T_2^* \supset -T_1$

integration by parts for AC functions

$$f, g \in D(T_1) \Rightarrow \int_0^\infty f'\bar g = \lim_{r\to\infty} \int_0^r f'\bar g \overset{\downarrow}{=} \lim_{r\to\infty}\left([f\bar g]_0^r - \int_0^r f\bar g'\right)$$

$$= \lim_{r\to\infty}\left(\underbrace{f(r)\overline{g(r)}}_{\to 0 \text{ for } r\to\infty \text{ by } (2)} - f(0)\overline{g(0)} - \int_0^r f\bar g'\right)$$

$$= -f(0)\overline{g(0)} - \int_0^\infty f\bar g'$$

hence:

$$(\square)\quad \forall f, g \in D(T_1) : \int_0^\infty f'\bar g = -f(0)\overline{g(0)} - \int_0^\infty f\bar g'$$

If f or g belongs to $D(T_2)$, then $f(0)\overline{g(0)} = 0$.

So: $f \in D(T_1), g \in D(T_2) \Rightarrow \langle T_1 f, g\rangle = -\langle f, T_2 g\rangle$

$\Rightarrow -T_2 \subset T_1^*$

$f \in D(T_2), g \in D(T_1) \Rightarrow \langle T_2 f, g\rangle = -\langle f, T_1 g\rangle$

$\Rightarrow -T_1 \subset T_2^*$

(4) $D(T_j^*) \subset D(T_1)$, $j = 1, 2$

$g \in D(T_j^*) \Rightarrow \exists\, h \in L^2(0,\infty)\ \forall f \in D(T_j): \int_0^\infty f' g = \int_0^\infty f h$

Define $H(x) = \int_0^x h$, $x \in [0,\infty)$. Then $H \in AC_{loc}[0,\infty)$, $H(0)=0$, $H' = h$

Thus $\forall f \in D(T_j)$: $\int_0^\infty f'\bar g = \int_0^\infty f\bar h = \lim_{R\to\infty} \int_0^R f\bar h = \lim_{R\to\infty} \Big(\underbrace{[f\bar H]_0^R}_{= f(R)\overline{H(R)} - f(0)\underbrace{\overline{H(0)}}_{=0}} - \int_0^R f'\bar H \Big)$

$= \lim_{R\to\infty} \Big(f(R)\overline{H(R)} - \int_0^R f'\bar H \Big)$

So, $\forall f \in D(T_j)$: $\int_0^\infty f'\bar g = \lim_{R\to\infty} \Big(f(R)\overline{H(R)} - \int_0^R f'\bar H \Big) \qquad (\lozenge)$

Since $D(T_j) \supset \mathcal{D}(0,\infty)$, we get

$\forall \varphi \in \mathcal{D}((0,\infty))$: $\int_0^\infty \varphi'\bar g = \lim_{R\to\infty} \Big(\underbrace{\varphi(R)\overline{H(R)}}_{=0 \text{ for } R \text{ large enough}} - \int_0^R \varphi'\bar H \Big)$

$= -\int_0^\infty \varphi'\bar H$

Hence $\forall \varphi \in \mathcal{D}((0,\infty))$: $\int_0^\infty \varphi' \cdot (\bar g + \bar H) = 0$.

So, $\bar g + \bar H$ is constant (by Prop VII.3)

Hence $\exists\, C \in \mathbb{K}$: $\bar g + \bar H = C$, thus $\bar g = C - \bar H$. In particular, $\bar g \in AC_{loc}[0,\infty)$. We know $g \in L^2(0,\infty)$ and $g' = -H' = -h \in L^2(0,\infty)$

$\Rightarrow g \in D(T_1)$

(5) By (4) and (3) we deduce $T_2^* = -T_1$

(6) $T_1^* = -T_2$: $g \in D(T_1^*) \overset{(4)}{\Longrightarrow} g \in D(T_1)$.

By (4) we know that $\bar g = C - \bar H$, clearly $C = \overline{g(0)}$ (as $H(0)=0$)

so, $\bar H = \overline{g(0)} - \bar g$

Plug this to (Δ):

$$g \in D(T_1^*) \Rightarrow \forall f \in D(T_1): \int_0^\infty f' \bar g = \lim_{r\to\infty} \left(f(r)\left(\overline{g(0)} - \overline{g(r)}\right) - \int_0^r f'\left(\overline{g(0)} - \bar g\right) \right)$$

$$= \lim_{r\to\infty} \Big(\underbrace{f(r)\overline{g(0)}}_{\to 0 \text{ by } (2)} - \underbrace{f(r)}_{\downarrow 0,\ (2)} \underbrace{\overline{g(r)}}_{\downarrow 0,\ (2)+(4)} - \underbrace{\int_0^r f' \, \overline{g(0)}}_{\substack{(f(r)-f(0))\overline{g(0)} \\ \downarrow \\ 0 \text{ by } (2)}} + \underbrace{\int_0^r f' \bar g}_{\to \int_0^\infty f' \bar g} \Big)$$

$$= f(0)\overline{g(0)} + \int_0^\infty f' \bar g$$

So, $\forall f \in D(T_1): f(0)\overline{g(0)} = 0$. Since there is $f \in D(T_1)$ with $f(0) \neq 0$, we deduce $g(0) = 0$, So $g \in D(T_2)$.

(7) We have proved $T_1^* = -T_2$, $T_2^* = -T_1$, hence T_1, T_2 are closed and iT_2 is symmetric

(8) Eigenvalues: $T_j f = \lambda f \Rightarrow f' = \lambda f \Rightarrow f(t) = c\, e^{\lambda t}$

- T_1: $t \mapsto e^{\lambda t}$ belongs to $D(T_1) \iff \mathrm{Re}\,\lambda < 0$
  So, $\sigma_p(T_1) = \{\lambda \in \mathbb{C}, \mathrm{Re}\,\lambda < 0\}$
- T_2: the initial condition $f(0)=0$ leads to only trivial solution. So, $\sigma_p(T_2) = \emptyset$

(9) Spectrum:

- $\sigma(T_1) \supset \overline{\sigma_p(T_1)} = \{\lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda \le 0\}$
- $\sigma(T_2) = \{\bar{\lambda};\ \lambda \in \sigma(T_2^*)\} = \{-\bar{\lambda},\ \lambda \in \sigma(T_1)\}$
  $\supset \{\lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda \ge 0\}$

iT_2 is symmetric, $\sigma(iT_2) = i\,\sigma(T_2) \supset \{\lambda \in \mathbb{C};\ \mathrm{Im}\,\lambda \ge 0\}$

Assume $\mathrm{Im}\,\lambda < 0$. Then $\lambda I - iT_2$ is one-to-one and has closed range (by Lemma XII.24).
Further,

$$R(\lambda I - iT_2)^\perp \underset{\text{Prop. XII.18}}{=} \ker(\bar{\lambda} I + i T_2^*) = \ker(\bar{\lambda} I - i T_1)$$

$$\ker(-i\bar{\lambda} I - T_1) = \{0\}$$

$\mathrm{Im}\,\lambda < 0 \Rightarrow \mathrm{Im}\,\bar{\lambda} > 0 \Rightarrow$

$\mathrm{Re}\,(-i\bar{\lambda}) > 0 \Rightarrow -i\bar{\lambda} \notin \sigma_p(T_1)$

So, $\lambda I - iT_2$ has dense range
So it is one-to-one and onto, therefore $\lambda \notin \sigma(iT_2)$

Conclusion $\sigma(iT_2) = \{\lambda \in \mathbb{C};\ \mathrm{Im}\,\lambda \ge 0\}$
$\Rightarrow \sigma(T_2) = \{\lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda \ge 0\}$

Thus $\sigma(T_1) = \{-\bar{\lambda};\ \lambda \in \sigma(T_2)\} = \{\lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda \le 0\}$

$\boxed{C}$ let T be the operator on $L^2(\mathbb{R})$ defined by

$$Tf = f',$$

$$f \in D(T) = \{ f \in AC_{loc}(\mathbb{R}) ; \; f, f' \in L^2(\mathbb{R}) \}$$

① T is densely defined as $\mathcal{D}(\mathbb{R}) \subset D(T)$ and $\mathcal{D}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ by Lemma VII-1

② $f \in D(T) \Rightarrow \lim\limits_{x \to \pm\infty} f(x) = 0$

$\ulcorner$ see $\boxed{B}$ ② $\lrcorner$

③ $T^* \supset -T$:

$f, g \in D(T)$

integrating by parts for AC functions

$$\langle Tf, g \rangle = \int_{-\infty}^{\infty} f' \bar{g} = \lim_{r \to +\infty} \int_{-r}^{r} f' \bar{g} = \lim_{r \to +\infty} \left([f \bar{g}]_{-r}^{r} - \int_{-r}^{r} f \bar{g}' \right)$$

$$= \lim_{r \to +\infty} \left(\underbrace{f(r)\bar{g}(r) - f(-r)\overline{g(-r)}}_{\to 0 \text{ by } ②} - \int_{-r}^{r} f \bar{g}' \right)$$

$$= -\int_{-\infty}^{\infty} f \bar{g}' = \langle f, -Tg \rangle$$

④ $T^* \subset -T$

$g \in D(T^*) \Rightarrow \exists \, h \in L^2(\mathbb{R}) \; \forall f \in D(T) : \; \langle Tf, g \rangle = \langle f, h \rangle$

$H(t) = \int_0^t h, \; t \in \mathbb{R}$. Then $H \in AC_{loc}(\mathbb{R})$, $H' = h$, $H(0) = 0$

Thus $\forall f \in D(T)$: $\displaystyle\int_{-\infty}^{\infty} f' \bar{g} = \int_{-\infty}^{\infty} f \bar{h} = \lim_{r \to +\infty} \int_{-r}^{r} f \bar{h} =$

$$= \lim_{r \to \infty} \left([f \bar{H}]_{-r}^{r} - \int_{-r}^{r} f' \bar{H} \right) = \lim_{r \to \infty} \left(f(r)\overline{H(r)} - f(-r)\overline{H(-r)} - \int_{-r}^{r} f' \bar{H} \right)$$

Since $D(T) \supset \mathcal{D}(\mathbb{R})$, we have

$$\forall \varphi \in \mathcal{D}(\mathbb{R}): \int_{-\infty}^{\infty} \varphi' \bar{g} = \lim_{r \to \infty} \Big(\underbrace{\varphi(r)\overline{H(r)} - \varphi(-r)\overline{H(-r)}}_{=0 \text{ for } r \text{ large enough}} - \int_{-r}^{r} \varphi' \bar{H} \Big)$$

$$= -\int_{-\infty}^{\infty} \varphi' \bar{H}. \quad \text{So } \forall \varphi \in \mathcal{D}(\mathbb{R}): \int_{-\infty}^{\infty} \varphi' (\bar{H} + \bar{g}) = 0$$

So, $H+g$ is constant (by Prop. VII.3)

$\Rightarrow \exists c \in \mathbb{C}: \quad g = c - H.$ Thus $g \in AC_{loc}(\mathbb{R})$

Further, $g \in L^2(\mathbb{R})$ and $g' = -h \in L^2(\mathbb{R})$

So, $g \in D(T)$.

(5) So, $T^* = -T$. In particular, T is closed and iT is self-adjoint.

(6) Eigenvalues: $Tf = \lambda f \Rightarrow f = \lambda f' \Rightarrow f = ce^{\lambda t}$.

But $t \mapsto e^{\lambda t}$ never belongs to $L^2(\mathbb{R})$

So, $\sigma_p(T) = \emptyset$

(7) Spectrum: iT self-adjoint $\Rightarrow \sigma(iT) \subset \mathbb{R} \Rightarrow \sigma(T) \subset i\mathbb{R}$

Take $\lambda \in \mathbb{R}$ and consider $i\lambda \mathbb{I} - T$. Is it onto?

$g \in L^2(\mathbb{R})$:

$$i\lambda f - f' = g$$

$$i\lambda f(t)e^{-i\lambda t} - f'(t)e^{-i\lambda t} = g(t)e^{-i\lambda t}$$

$$(-f(t)e^{-i\lambda t})' = g(t)e^{-i\lambda t}$$

$$-f(t)e^{-i\lambda t}+f(0)=\int_0^t g(s)e^{-i\lambda s}\,ds$$

$$f(t)=e^{i\lambda t}\left(f(0)-\int_0^t g(s)e^{-i\lambda s}\,ds\right)$$

$$f\in D(T)\overset{(2)}{\Longrightarrow}\lim_{t\to\pm\infty}f(t)=0$$

$$\Rightarrow\quad f(0)=\lim_{t\to+\infty}\int_0^t g(s)e^{-i\lambda s}\,ds$$

$$=\lim_{t\to-\infty}-\int_t^0 g(s)\,e^{-i\lambda s}\,ds$$

There are $g\in L^2(\mathbb{R})$ for which the equality fails, for example $g=\mathbb{1}_{(0,r)}$ for suitable $r>0$

Then the second limit is 0 and the first one $\int_0^r e^{-i\lambda s}ds=\left[\frac{e^{-i\lambda s}}{-i\lambda}\right]_0^r$

$$=\frac{e^{-i\lambda r}-1}{-i\lambda}\neq 0 \text{ if } \lambda r \text{ is not a multiple of } 2\pi$$

(if $\lambda\neq0$;
if $\lambda=0$, then the integral $=r$)

Conclusion: $\sigma(T)=i\mathbb{R}$