

Proposition XII. 5 H -- a Hilbert space, $T \in \mathcal{L}(H)$

$$(a) T^* = T \Leftrightarrow \mathcal{W}(T) \subset \mathbb{R}$$

$$T^* = T \stackrel{\text{Prop. 3(c)}}{\Leftrightarrow} \forall x \in H : \langle Tx, x \rangle = \langle T^*x, x \rangle$$

$$\Leftrightarrow \forall x \in H : \langle Tx, x \rangle \in \mathbb{R} \Leftrightarrow \mathcal{W}(T) \subset \mathbb{R}$$

$$(b) \text{ Suppose } T^* = T. \quad a := \inf \mathcal{W}(T), \quad b = \sup \mathcal{W}(T).$$

$$\text{Then } \sigma(T) \subset [a, b] \quad (\text{by Prop. 3(e)}, \text{ as } [a, b] \supset \overline{\mathcal{W}(T)})$$

$$\text{In fact } [a, b] = \overline{\mathcal{W}(T)} \quad \text{by Prop. 3(d)}$$

$$\bullet T \text{ self-adjoint} \Rightarrow \mathcal{R}(T) = \|T\|. \quad \text{Hence, by Prop. 3(gf)}$$

$$\text{we get } \mathcal{R}(T) = \mathcal{W}(T) = \|T\|$$

$$\text{Since } \mathcal{W}(T) = \max \{|a|, |b|\} \quad (= \max(b, -a))$$

we get

$$\mathcal{R}(T) = \|T\| = \max \{|a|, |b|\}$$

$$\bullet \text{ Since } \sigma(T) \text{ is compact, } \sigma(T) \subset [a, b] \text{ and}$$

$$\mathcal{R}(T) = \|T\| = \max \{|a|, |b|\}, \text{ necessarily } \|T\| \in \sigma(T)$$

$$\text{or } -\|T\| \in \sigma(T)$$

$$\bullet \text{ In fact } a, b \in \sigma(T) :$$

$$S_1 := T - aI, \quad S_2 := T - bI$$

$$\text{Then } S_1, S_2 \text{ are self-adjoint, } \overline{\mathcal{W}(S_1)} = [0, b-a]$$

$$\overline{\mathcal{W}(S_2)} = [a-b, 0]. \quad \text{Thus } b-a \in \sigma(S_1), \quad a-b \in \sigma(S_2)$$

$$\Rightarrow b \in \sigma(T), \quad a \in \sigma(T).$$

$$(c) \quad W(T) \subset [0, \infty) \Leftrightarrow T^* = T \text{ and } \sigma(T) \subset [0, \infty)$$

\Rightarrow by (a) and Prop 3(e) $S = (T)W \Leftrightarrow T = T^*$ (B)

\Leftarrow Define a, b as in (b). Then $a, b \in \sigma(T)$. So $a \geq 0$
Thus $W(T) \subset [0, \infty)$.

$$\overline{S + T} = S + T + S$$

$$[S + T](T)W \subset S + T + T^*W \subset H \Rightarrow$$

$$(T)W \text{ and } S = (T)W \Rightarrow S + T + T^*W \subset H \Rightarrow T = T^*$$

$$[2, \infty) \supset (T)W \Rightarrow (T)W = [2, \infty)$$

$$\overline{(T)W} \subset [2, \infty)$$

$$\text{and } (T)W \subset [2, \infty)$$

$$(T)W = [2, \infty)$$

$$(12) \text{ and } (12)^* = (12) \Rightarrow \text{No } \sigma_{\text{ess}}(T) \text{ in } [2, \infty)$$

$$\|T\| = (T)W = (T)^* \Rightarrow \text{No } \sigma_{\text{ess}}$$

$$(12 - \lambda) \text{ and } (12 - \lambda)^* = (12) - \lambda \text{ and } \lambda \in \sigma_{\text{ess}}$$

$$\text{No } \sigma_{\text{ess}}$$

$$\{12, 101\} \text{ and } \|T\| = (T)W = (T)^*$$

$$\text{and } [2, \infty) \supset (T)W \text{ and } (T)W \text{ and } (T)^*W$$

$$(T)W \neq \emptyset \text{ and } \{12, 101\} \text{ and } \|T\| = (T)W = (T)^*$$

$$(T)W \neq \emptyset \text{ and } (T)^*W \neq \emptyset$$

$$(T)W \neq \emptyset \text{ and } (T)^*W \neq \emptyset$$

$$12 - T = \pm 2, \quad 101 - T = \pm 2$$

$$\{101, 12\} = \overline{(2)W}, \text{ and } \sigma_{\text{ess}}(T) \text{ in } \{2, 12\} \text{ and } T$$

$$(2)W \neq \emptyset, (2)W \neq \emptyset \text{ and } \sigma_{\text{ess}}(T) \text{ in } \{12, 101\} = \overline{(2)W}$$

$$(T)W \neq \emptyset, (T)W \neq \emptyset \text{ and } (T)^*W \neq \emptyset$$