

Lemma XII.2

(Polarization formula for operators)

 H -- a Hilbert space, $T \in \mathcal{L}(H)$, $x, y \in H$:

$$\langle T(x+y), x+y \rangle = \langle Tx, x \rangle + \langle Tx, y \rangle + \langle Ty, x \rangle + \langle Ty, y \rangle$$

$$\langle T(x-y), x-y \rangle = \langle Tx, x \rangle - \langle Tx, y \rangle - \langle Ty, x \rangle + \langle Ty, y \rangle$$

Subtract:

$$(*) \quad \langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle = 2(\langle Tx, y \rangle + \langle Ty, x \rangle)$$

Apply this formula to x and iy :

$$\begin{aligned} \langle T(x+iy), x+iy \rangle - \langle T(x-iy), x-iy \rangle &= 2(\langle Tx, iy \rangle + \langle T(iy), x \rangle) = \\ &= 2i(-\langle Tx, y \rangle + \langle Ty, x \rangle) \end{aligned}$$

$$\text{So, } i \langle T(x+iy), x+iy \rangle - i \langle T(x-iy), x-iy \rangle = 2(\langle Tx, y \rangle - \langle Ty, x \rangle) \quad (**)$$

Add (*) and (**):

$$\begin{aligned} \langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle + i \langle T(x+iy), x+iy \rangle - i \langle T(x-iy), x-iy \rangle \\ \parallel \\ 4 \langle Tx, y \rangle. \end{aligned}$$

Proposition XII.3 (properties of the numerical radius)

H -- a Hilbert space

For $T \in \mathcal{L}(H)$ set

$$W(T) = \{ \langle Tx, x \rangle ; x \in S_H \} \quad (\text{numerical range})$$

$$w(T) = \sup \{ |\lambda| ; \lambda \in W(T) \} \quad (\text{numerical radius})$$

(a) w is an equivalent norm, $\frac{1}{2} \|T\| \leq w(T) \leq \|T\|$ for $T \in \mathcal{L}(H)$

Clearly, $w(T) \geq 0$, $w(0) = 0$

- $x \in S_H \Rightarrow |\langle Tx, x \rangle| \leq \|Tx\| \cdot \|x\| \leq \|T\|$,
so $w(T) \leq \|T\|$, in fact $w(T) \in [0, \infty)$
- $w(\alpha T) = |\alpha| w(T) \quad T \in \mathcal{L}(H), \alpha \in \mathbb{C}$
 $w(S+T) \leq w(S) + w(T) \quad S, T \in \mathcal{L}(H)$

So, w is a seminorm. It remains to prove $\frac{1}{2} \|T\| \leq w(T)$

For $x, y \in S_H$ we have, due to Lemma 2

$$|\langle Tx, y \rangle| \leq \frac{1}{4} (|\langle T(x+y), x+y \rangle| + |\langle T(x-y), x-y \rangle| + |\langle T(x+iy), x+iy \rangle| + |\langle T(x-iy), x-iy \rangle|) \quad (*)$$

$$\uparrow |\langle T(x+y), x+y \rangle| \leq w(T) \cdot \|x+y\|^2$$

• if $x+y=0$, obvious

$$\cdot x+y \neq 0 \quad \rightarrow \quad |\langle T(x+y), x+y \rangle| = \underbrace{|\langle T(\frac{x+y}{\|x+y\|}, \frac{x+y}{\|x+y\|}) \rangle|}_{\leq w(T)} \|x+y\|^2 \quad \rightarrow$$

$$(*) \leq \frac{1}{4} w(T) (\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2) =$$

$$= \frac{1}{4} w(T) (2(\|x\|^2 + \|y\|^2) + 2(\|x\|^2 + \|y\|^2)) = 2w(T)$$

\uparrow the parallelogram rule

Thus $|\langle Tx, x \rangle| \leq 2\omega(T) \quad x \in S_H$

It follows $\|T\| \leq 2\omega(T)$ ┘

(b) $\langle Tx, x \rangle = 0$ for $x \in H \Rightarrow T = 0$

$\Gamma \forall x \in H: \langle Tx, x \rangle = 0 \Rightarrow \omega(T) = 0 \xrightarrow{(a)} \|T\| = 0 \Rightarrow T = 0$ ┘

(c) $\langle Tx, x \rangle = \langle Sx, x \rangle$ for $x \in H \Rightarrow T = S$

Γ Apply (b) to $T - S$ ┘

(d) $\omega(T)$ is a connected subset of \mathbb{C} for $T \in \mathcal{L}(H)$

ΓS_H is connected and $x \mapsto \langle Tx, x \rangle$ is c.t.s. ┘

(e) $\sigma_p(T) \subset \omega(T)$, $\overline{\sigma(T)} \subset \overline{\omega(T)}$

$\Gamma \lambda \in \sigma_p(T) \Rightarrow \exists x \in S_H \quad \lambda x = Tx \Rightarrow \langle Tx, x \rangle = \langle \lambda x, x \rangle = \lambda \Rightarrow \lambda \in \omega(T)$

$\lambda \in \sigma(T) \Rightarrow$ Prop 1 (c) $\lambda \in \sigma_{ap}(T)$ or $\lambda \in \sigma_r(T)$

$\lambda \in \sigma_{ap}(T) \Rightarrow \exists (x_n) \subset S_H \quad \|\lambda x_n - Tx_n\| \rightarrow 0$

So $\langle \lambda x_n - Tx_n, x_n \rangle \rightarrow 0 \quad \left. \begin{array}{l} \Rightarrow \langle \lambda x_n, x_n \rangle \rightarrow \lambda \\ \lambda - \langle Tx_n, x_n \rangle \end{array} \right\} \Rightarrow \lambda \in \overline{\omega(T)}$

$\lambda \in \sigma_r(T) \Rightarrow R(\lambda I - T)$ is not dens $\Rightarrow \exists x \in S_H \quad x \perp R(\lambda I - T)$

So, in particular $\langle \lambda x - Tx, x \rangle = 0$, so $\langle Tx, x \rangle = \lambda \Rightarrow \lambda \in \omega(T)$ ┘

(f) $\omega(T) \supseteq \sigma(T)$ ┘
 Γ follows from (e) ┘