Proof of Prop. XII.1 : X — a Banach space, $T \in \mathcal{L}(X)$

(a) $\sigma_p(T) \subset \sigma_{ap}(T)$

$\lceil$ obvious: $\lambda \in \sigma_p(T) \Rightarrow \exists x \in X \; \|x\|=1 \; Tx = \lambda x$

Take $x_n = x$ for $n \in \mathbb{N}$. Then $(Tx_n - \lambda x_n) = 0 \to 0$,

so $\lambda \in \sigma_{ap}(T)$ $\rfloor$

(b) $\lambda \in \mathbb{C} \setminus \sigma_{ap}(T) \iff \lambda I - T$ is an isomorphism of X into X

$\lceil \lambda \in \mathbb{C} \setminus \sigma_{ap}(T) \iff \exists c > 0 \; \forall x \in S_X : \|(\lambda I - T)x\| \geq c$

$\iff \exists c > 0 \; \forall x \in X : \|(\lambda I - T)x\| \geq c\|x\|$

$\iff \lambda I - T$ is an into isomorphism $\rfloor$

(c) $\sigma(T) = \sigma_{ap}(T) \cup \sigma_r(T)$

$\lceil \supset$ obvious.

$\subset : \lambda \notin \sigma_{ap}(T) \cup \sigma_r(T) \overset{(b)}{\Rightarrow} \lambda I - T$ is an isomorphism of X into X,
hence its range is closed.

In part. $\lambda I - T$ is one-to-one, hence $R(\lambda I - T)$ is dense
(as $\lambda \notin \sigma_r(T)$).

Thus $R(\lambda I - T) = X$ (being closed and dense),
thus $\lambda I - T$ is invertible. So $\lambda \notin \sigma(T)$ $\rfloor$

(d) $\sigma_c(T) = \sigma_{ap}(T) \setminus (\sigma_p(T) \cup \sigma_r(T)) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_r(T))$

$\lceil$ ① $\lambda \in \sigma_c(T) \Rightarrow \quad \bullet \; \lambda I - T$ is one-to-one, hence $\lambda \notin \sigma_p(T)$

$\bullet \; R(\lambda I - T)$ is dense, hence $\lambda \notin \sigma_r(T)$

$\bullet \; \lambda \in \sigma(T) \setminus \sigma_r(T) \Rightarrow \lambda \in \sigma_{ap}(T)$ by (c)

② $\lambda \in \sigma(T) \setminus (\sigma_p(T) \cup \sigma_r(T)) \Rightarrow \quad \bullet \; \lambda \notin \sigma_p(T) \Rightarrow \lambda I - T$ is one-to-one

As $\lambda \in \sigma(T)$, $\lambda I - T$ is not onto

$\bullet \; \lambda \notin \sigma_r(T)$, $\lambda I - T$ one-to-one $\Rightarrow$
$R(\lambda I - T)$ is dense.

Hence $\lambda \in \sigma_c(T)$. $\rfloor$

(e) $\lambda \in \sigma_r(T) \setminus \sigma_{ap}(T) \iff \lambda I - T$ is an isomorphism of X onto a proper closed subspace of X

$\Longrightarrow$ $\lambda \in \sigma_r(T) \setminus \sigma_{ap}(T) \Rightarrow$
- $\lambda \in \mathbb{C} \setminus \sigma_{ap}(T) \overset{(b)}{\Longrightarrow} \lambda I - T$ is an isomorphism of X into X.
  In part. $R(\lambda I - T)$ is closed and $\lambda I - T$ is one-to-one
- $\lambda \in \sigma_r(T)$,
  $\Rightarrow R(\lambda I - T)$ is not dense

Hence $R(\lambda I - T)$ is a proper closed subspace of X

$\Longleftarrow$ $\lambda I - T$ is an isomorphism of X into $X \overset{(b)}{\Rightarrow} \lambda \in \mathbb{C} \setminus \sigma_{ap}(T)$

Moreover, $\lambda I - T$ is one-to-one and $R(\lambda I - T)$ is not dense, so $\lambda \in \sigma_r(T)$