Proof of Proposition XI.18

H, K Hilbert spaces, $T \in \mathcal{L}(H, K)$

(i) $\Rightarrow$ (ii) Suppose T is unitary, i.e. $T^* = T^{-1}$.
Then • T is an isometry
$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle \underset{\uparrow\, T^*T = I_H}{=} \langle x, x\rangle = \|x\|^2$$

• T is onto as it has an inverse

(ii) $\Rightarrow$ (iii) T is an onto isometry $\Rightarrow$ T is an into isometry
[trivial]

(iii) $\Rightarrow$ (iv) Suppose T is an isometry of H into K.
Then for $x, y \in H$ we have
$$\langle Tx, Ty\rangle_K \underset{\uparrow}{=} \frac{1}{4}\left(\|Tx+Ty\|^2 - \|Tx-Ty\|^2 + i\|Tx+iTy\|^2 - i\|Tx-iTy\|^2\right)$$
polarization identity (Prop. I.21)
$$= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right) = \langle x, y\rangle_H$$
T is an isometry

(iv) $\Rightarrow$ (iii) Suppose $\langle Tx, Ty\rangle_K = \langle x, y\rangle_H$, $x, y \in H$.
Apply for $y = x$ and deduce that T is an isometry

(iii) $\Rightarrow$ (ii) T is an into isometry & T is onto $\Rightarrow$ T is an onto isome-
[if T is onto] [trivial]

(ii) $\Rightarrow$ (i) Suppose T is an isometry of H onto K. Then T^{-1} exis
Moreover, by the already proved (iii) $\Rightarrow$ (iv) we have
$$\langle Tx, Ty\rangle_K = \langle x, y\rangle_H \text{ for } x, y \in H$$

Thus for $x \in H, y \in K$ we have
$$\langle Tx, y\rangle_K = \langle Tx, TT^{-1}y\rangle_K = \langle x, T^{-1}y\rangle_H. \quad \text{Thus } T^{-1} = T^*$$