Theorem XI.15

Let A be a C^*-algebra, unital or not, $x \in A$ normal element
$B := \overline{\mathrm{alg}}^{\|\cdot\|}\{x, x^*\}$

Then B is a commutative C^*-subalgebra of A.

Consider the C^*-algebra A^+ and set
$\tilde{B} = \overline{\mathrm{alg}}^{\|\cdot\|}\{(x,0), (x^*,0), (0,1)\} = \{(b,\lambda),\ b \in B,\ \lambda \in \mathbb{C}\}$

Apply Theorem XI.14 to $\tilde{B}$:

$h^+ : \varphi \mapsto \varphi(x,0)$ is a homeomorphism of $\Delta(\tilde{B})$ onto $\sigma_{A^+}(x,0)$

Let $\Gamma^+ : \tilde{B} \to \mathcal{C}(\Delta(\tilde{B}))$ be the Gelfand transform

and $\Phi^+ : \mathcal{C}(\sigma(x,0)) \to \tilde{B}$ be defined

by $\Phi^+ : f \mapsto \tilde{f}^+(x) = (\Gamma^+)^{-1}(f \circ h^+)$

Then Φ^+ has the properties (a)–(e) from Thm XI.14

Further, let $h : \varphi \mapsto \varphi(x)$ $\quad (h : \Delta(B) \cup \{0\} \to \mathbb{C})$

Since $\theta : \Delta(\tilde{B}) \to \Delta(B) \cup \{0\}$
defined by $\theta(\varphi)(b) = \varphi(b,0)$, $b \in B$, $\varphi \in \Delta(\tilde{B})$
is a homeomorphism (Prop. X.23(b) implies it is a bijection, it is clearly cts),

$h = h^+ \circ \theta^{-1}$ is a homeomorphism of $\Delta(B) \cup \{0\}$
onto $\sigma_{A^+}(x,0) = \sigma_A(x) \cup \{0\}$

Let $\Gamma : B \to \mathcal{C}_0(\Delta(B))$ be the Gelfand transform and define $\Phi : \mathcal{C}_0(\sigma(x)\setminus\{0\}) \to B$

by $\Phi(f) = \tilde{f}(x) = \Gamma^{-1}(f \circ h)$.

$$\mathcal{C}_0(\sigma(x)\setminus\{0\}) \simeq \{ f \in \mathcal{C}(\sigma(x)\cup\{0\}),\ f(0)=0 \}$$

$$f \in \mathcal{C}(\sigma(x)\cup\{0\}) \Rightarrow f - f(0) \in \mathcal{C}_0(\sigma(x)\setminus\{0\})$$

Moreover, $\tilde{f}^+(x) = (\widetilde{f-f(0)}(x), f(0))$

To prove this, it is enough to check that

$$\Gamma^+(\widetilde{f-f(0)}(x), f(0)) = f \circ h^+$$

By Prop. X.23 (b) $\quad \Delta(\tilde{B}) = \{ \tilde{\varphi},\ \varphi \in \Delta(B)\} \cup \{\varphi_\infty\}$

$$\Gamma^+(\widetilde{f-f(0)}(x), f(0))(\tilde{\varphi}) \overset{\text{Thm X.25(5)}}{=} \Gamma(\widetilde{f-f(0)}(x))(\varphi) + f(0) =$$
$$= (f-f(0)) \circ h(\varphi) + f(0) = (f-f(0))(\varphi(x)) + f(0) = f(\varphi(x)) - f(0) + f(0)$$
$$= f(\varphi(x)) = f(\tilde{\varphi}(x,0)) = f \circ h^+(\tilde{\varphi})$$

$$\Gamma^+(\widetilde{f-f(0)}, f(0))(\varphi_\infty) \overset{\text{Thm 24 (5)}}{=} f(0) = f(\varphi_\infty(x,0)) = f \circ h^+(\varphi_\infty)$$

It follows that Φ^+ maps $\{ f \in \mathcal{C}(\sigma(x)\cup\{0\}),\ f(0)=0\}$

onto $\{(s,0),\ s \in B\}$. Thus Φ maps $\mathcal{C}_0(\sigma(x)\setminus\{0\})$

onto B. The properties (a)–(e) follow from the respective ones in Thm XI.14