A commutative C^*-algebra, $\Gamma : A \to \mathcal{C}_0(\Delta(A))$ its Gelfand transform.

Thm Γ is an isometric $*$-isomorphism of A onto $\mathcal{C}_0(\Delta(A))$

Proof:

(1) Thm X.25(g) $\Rightarrow \forall x \in A \quad \|\Gamma(x)\| = r(x)$
A commutative $\Rightarrow$ each $x \in A$ is normal, thus
$r(x) = \|x\|$ by Prop. XI.3

It follows that Γ is an isometry of A into $\mathcal{C}_0(\Delta(A))$

(2) Γ is a homomorphism by Thm X.25(a)
Further, for $x \in A$, $h \in \Delta(A)$ we have
$\Gamma(x^*)(h) = h(x^*) = \overline{h(x)} = \overline{\Gamma(x)(h)}$
$\uparrow$
Prop. XI.8(c)

So, Γ is a $*$-homomorphism.

(3) $\Gamma(A)$ is a subalgebra of $\mathcal{C}_0(\Delta(A))$ separating points of $\Delta(A)$ (Thm X.25(j)) and, moreover

$f \in \Gamma(A) \Rightarrow \bar{f} \in \Gamma(A)$ by (2). The Stone-Weierstrass theorem says that $\Gamma(A)$ is dense in $\mathcal{C}_0(\Delta(A))$

[more precisely: A unital $\Rightarrow 1 \in \Gamma(A)$, so
$\Gamma(A)$ contains constants,
so we can use S-W theorem,
for non-unital case see (4) below]

By (1) we get that $\Gamma(A)$ is closed in $\mathcal{C}_0(\Delta(A))$,
so $\Gamma(A) = \mathcal{C}(\Delta(A))$ if
A is unital

(4) A not unital, take $\Gamma^+ : A^+ \to \mathcal{C}(\Delta(A^+))$

we know already that Γ^+ is onto.

It follows from Thm X.25(5) that

$$\Gamma^+(\{(a,0),\ a\in A\}) = \{f \in \Gamma^+(A^+)\ ;\ f(h_\infty) = 0\}$$

So, $\Gamma(A) = \mathcal{C}_0(\Delta(A))$.

(5) It follows that A is unital if and only if $\Delta(A)$ is compact ($\Leftrightarrow$ $C_0(\Delta(A))$ is unital)

$\boxed{XI}$

Corollary XI.10 A, B commutative C^*-algebras
A, B are $*$-isomorphic $\Leftrightarrow$ $\Delta(A)$ and $\Delta(B)$ are homeomorphic

Proof: $\Rightarrow$: $T: A \to B$ $*$-isomorphism onto
By Prop. XI.6 $\|T\| \le 1$ and $\|T^{-1}\| \le 1$,
so T is an isometry
Thus $T': B^* \to A^*$ is a w^*-w^* homeomorphism
Moreover, $T'(\Delta(B)) = \Delta(A)$ as
T is an isomorphism of Banach algebras

$\Leftarrow$ $\Delta(A)$ homeomorphic to $\Delta(B)$ $\Rightarrow$ $\mathcal{C}_0(\Delta(A))$ is $*$-isomorphic to $\mathcal{C}_0(\Delta(B))$

Hence $A \approx \mathcal{C}_0(\Delta(A))$ is $*$-isomorphic

to $B \approx \mathcal{C}_0(\Delta(B))$