EXAMPLE XI.7

Let K, L be compact Hausdorff spaces
$\varphi: \mathcal{C}(K) \to \mathcal{C}(L)$ be a $*$-homomorphism, $\varphi(1)=1$
Then $\exists\ \alpha: L \to K$ cts s.t. $\varphi(f) = f\circ\alpha$, $f \in \mathcal{C}(K)$

By Prop. XI.6 we know $\|\varphi\| \leq 1$, thus φ is cts.

Thus $\varphi': \mathcal{C}(L)^* \to \mathcal{C}(K)^*$ is well defined.

We have $\varphi'(\Delta(\mathcal{C}(L))) \subset \Delta(\mathcal{C}(K))$

$h \in \Delta(\mathcal{C}(L)) \Rightarrow$ $\varphi'(h)(1) = h(\varphi(1)) = h(1) = 1$
$\varphi'(h)(fg) = h(\varphi(fg)) = h(\varphi(f)\varphi(g)) =$
$= h(\varphi(f))\, h(\varphi(g)) = \varphi'(h)(f)\, \varphi'(h)(g)$

For $x \in L$ let $\delta_x \in \mathcal{C}(L)^*$ be the evaluation map
$\delta_x(f) = f(x)$, $f \in \mathcal{C}(L)$

Then $\delta_x \in \Delta(\mathcal{C}(L))$, thus $\varphi'(\delta_x) \in \Delta(\mathcal{C}(K))$.
Therefore $\exists!\ \alpha(x) \in K$ s.t. $\varphi'(\delta_x) = \delta_{\alpha(x)}$.

Thus we have a mapping $\alpha: L \to K$

For $f \in \mathcal{C}(K)$ and $x \in L$ we have
$\varphi(f)(x) = \delta_x(\varphi(f)) = \varphi'(\delta_x)(f) = \delta_{\alpha(x)}(f) = f(\alpha(x))$

So, $\varphi(f) = f\circ\alpha$.

It remains to show that α is cts.

Observe: $x \mapsto \delta_x$ is cts $L \to (\mathcal{C}(L)^*, w^*)$

[indeed, $\forall f \in \mathcal{C}(L)$ $x \mapsto \delta_x(f) = f(x)$ is cts]
and use Prop. VI.1(6)

$x \mapsto \delta_x$ is one-to-one
[$x \neq y \Rightarrow \exists f \in C(L)$ $f(x) \neq f(y)$
by Urysohn lemma.]

So, $x \mapsto \delta_x$ is a homeomorphism (as L is compact Hausdorff)

Further, φ' is w^*-to-w^* cts, as any dual operator

[$h \in C(L)^* \mapsto \varphi'(h)$ is w^*-w^*-cts,
as for each $f \in C(K)$
$h \mapsto \varphi'(h)(f) = h(\varphi(f))$ is w^*-cts] again Prop. VI.1(b)

Thus $\alpha : x \longmapsto \delta_x \longmapsto \varphi'(\delta_x) = \delta_{\alpha(x)} \longmapsto \alpha(x)$

is cts, as the composition of three cts maps.]

If φ is one-to-one, then $\alpha(L) = K$, hence φ is an isometry

[$\alpha(L) \subset K$ is closed (α cts, K, L compact Hausdorff)
$\Rightarrow$ if $x \in K \setminus \alpha(L) \Rightarrow \exists f \in C(K)$ s.t. $f(x) = 1$, $f|_{\alpha(L)} = 0$
(by Urysohn lemma)
Then $f \neq 0$ and $\varphi(f) = 0$, so φ is not one-to-one.

If $\alpha(L) = K$, then

$$\|\varphi(f)\|_{C(L)} = \sup_{x \in L} |\varphi(f)(x)| = \sup_{x \in L} |f(\alpha(x))| =$$

$$= \sup_{y \in \alpha(L)} |f(y)| = \sup_{y \in K} |f(y)| = \|f\|_{C(K)}.]$$