GELFAND TRANSFORM

Let A be a commutative Banach algebra

① For $x \in A$ define $\hat{x} : \Delta(A) \to \mathbb{C}$ by
$\hat{x}(h) = h(x)$, $h \in \Delta(A)$

Then $\hat{x}$ is cts on $\Delta(A)$ (by the definition of the w^*-topology)

So, if A is unital, then $\Delta(A)$ is compact, $\hat{x} \in C(\Delta(A)) = C_0(\Delta(A))$

If A is not unital, then $\Delta(A) \cup \{0\}$ is compact and if we extend $\hat{x}$ to $\Delta(A) \cup \{0\}$ by $\hat{x}(0) = 0$, it will be still cts
So, $\hat{x} \in C_0(\Delta(A)) \simeq \{f \in C(\Delta(A) \cup \{0\}) ; f(0) = 0\}$

② Define $\Gamma : A \to C_0(\Delta(A))$ by $\Gamma(x) := \hat{x}$, $x \in A$.
Γ is the Gelfand transform of A

③ We prove (a): Γ is a homomorphism of A into $C_0(\Delta(A))$

Γ Γ linear: $\Gamma(\alpha x + \beta y)(h) = h(\alpha x + \beta y) = \alpha h(x) + \beta h(y) =$
$= \alpha \Gamma(x)(h) + \beta \Gamma(y)(h)$
Γ multiplicative: $\Gamma(xy)(h) = h(xy) = h(x) h(y) = \Gamma(x)(h)\Gamma(y)(h)$ $\rfloor$

④ We prove (b): Let $\Gamma^+ : A^+ \to C(\Delta(A^+))$ be the G.T. of A^+.

By Prop. 23 (b) $\Delta(A^+) = \{\tilde{h} ; h \in \Delta(A)\} \cup \{h_\infty\}$, where
$\tilde{h}(x, \lambda) = h(x) + \lambda$, $h_\infty(x, \lambda) = \lambda$ $(x, \lambda) \in A^+$

Then $\Gamma^+(x, \lambda)(\tilde{h}) = \tilde{h}(x, \lambda) = h(x) + \lambda = \Gamma(x)(h) + \lambda$

$\Gamma^+(x, \lambda)(h_\infty) = h_\infty(x, \lambda) = \lambda$

⑤ We prove (c): A unital $\Rightarrow \ker \Gamma = \bigcap\{I;\ I$ is a maximal ideal in $A\}$ $(=: \operatorname{rad} A)$

$\ulcorner$ $\ker \Gamma = \{x \in A;\ \hat{x} = 0\} = \bigcap_{h \in \Delta(A)} \ker h \underset{\text{Prop. 24 (2)}}{=} \bigcap\{I;\ I$ a maximal ideal in $A\}$ $\lrcorner$

So, Γ is one-to-one $\Leftrightarrow \operatorname{rad} A = \{0\}$

⑥ Γ is one-to-one $\Leftrightarrow \Gamma^+$ is one-to-one

$\ulcorner \Leftarrow$: Γ is not one-to-one $\Rightarrow \exists x \in A \setminus \{0\}$: $\Gamma(x) = 0$
Then $(x, 0) \in A^+ \setminus \{(0,0)\}$, $\Gamma^+(x, 0) = 0$ by (3). Thus Γ^+ is not one-to-one

$\Rightarrow$) Γ^+ is not one-to-one $\Rightarrow \exists (x, \lambda) \in A^+ \setminus \{(0,0)\}$ $\Gamma^+(x, \lambda) = 0$
Hence $\Gamma^+(x, \lambda)(h_\infty) = 0$, so $\lambda = 0$. Thus $x \neq 0$.

Further, for each $h \in \Delta(A)$:
$0 = \Gamma^+(x, 0)(\tilde{h}) = \hat{h}(x, 0) = h(x) = \Gamma(x)(h)$
So $\Gamma(x) = 0$. Thus Γ is not one-to-one. $\lrcorner$

⑦ A unital $\Rightarrow$ $\forall x \in A$: $\hat{x}(\Delta(A)) = \sigma(x)$

$\ulcorner$ $\hat{x}(\Delta(A)) \subset \sigma(x)$ by Prop. 22 (f)

Conversely: $\lambda \in \sigma(x) \Rightarrow y := (\lambda e - x)$ is not invertible
Let $yA = \{yz,\ z \in A\} \Rightarrow yA$ is an ideal in A
Choose $I \supset yA$ maximal ideal
By prop. 24 (2) $\exists h \in \Delta(A)$: $I = \ker h$
Then $h(y) = 0$, so $h(x) = \lambda$, i.e. $\hat{x}(h) = \lambda$ $\lrcorner$

(8) A not unital $\Rightarrow$ $\sigma(x) = \hat{x}(\Delta(A)) \cup \{0\}$

$\ulcorner$ $\sigma(x) = \sigma_{A^+}(x,0) = \widehat{(x,0)}(\Delta(A^+)) = (\circ)$

By (5): $\widehat{(x,0)}(\tilde{h}) = h(x) = \hat{x}(h)$

$\widehat{(x,0)}(h_\infty) = 0$

So, $(\circ) = \hat{x}(\Delta(A)) \cup \{0\}$. $\lrcorner$

(9) $\|\hat{x}\| = r(x)$, $x \in A$

$\ulcorner$ This follows by (7) and (8) (i.e., by (e),(f)) $\lrcorner$

(10) $\|\Gamma\| \leq 1$, hence Γ is a cts homomorphism

$\ulcorner$ By (9): $\|\Gamma(x)\| = \|\hat{x}\| = r(x) \leq \|x\|$ $\lrcorner$

(11) Γ is a topological isomorphism of A and $\Gamma(A)$

$\Leftrightarrow$ Γ is one-to-one and $\Gamma(A)$ is closed

$\ulcorner$ Use open mapping theorem $\lrcorner$

(12) $\Gamma(A)$ separates points of $\Delta(A)$

$\ulcorner$ $h_1, h_2 \in \Delta(A)$, $h_1 \neq h_2$ $\Rightarrow$ $\exists x \in A$ $h_1(x) \neq h_2(x)$.

So $\hat{x}(h_1) \neq \hat{x}(h_2)$. $\lrcorner$