

Proposition X.23

Let  $A$  be a Banach algebra

(a)  $A$  unital  $\Rightarrow \Delta(A)$  is a  $w^*$ -compact subset of  $S_{A^*}$ .

By Prop. 22 we have

$\Delta(A) \subset S_{A^*}$

$\Delta(A) = \{ h \in A^* ; \forall x, y \in A : h(x+y) = h(x) + h(y) \text{ \& } h(1) = 1 \}$ ,  
so  $\Delta(A)$  is  $w^*$ -closed.

(Corollary VI.16)

By Banach-Alaoglu  $B_{A^*}$  is  $w^*$ -compact,  
hence  $\Delta(A)$  is  $w^*$ -compact.  $\lrcorner$

(b)  $\Delta(A^+) = \{ \tilde{h} ; h \in \Delta(A) \} \cup \{ h_{\infty} \}$ , where  
 $\tilde{h}(x, \lambda) = h(x) + \lambda$ ,  $(x, \lambda) \in A^+$   
 $h_{\infty}(x, \lambda) = \lambda$ ,  $(x, \lambda) \in A^+$

" $\supset$ ": By Prop. 22  $\tilde{h} \in \Delta(A^+)$  whenever  $h \in \Delta(A)$   
Moreover, clearly  $h_{\infty} \in \Delta(A^+)$

" $\subset$ ": Let  $g \in \Delta(A^+)$ . Define  $h(x) = g(x, 0)$ ,  $x \in A$   
Then  $h \in A^*$ ,  $h$  is multiplicative  
 $\Rightarrow$  either  $h \in \Delta(A)$  (then  $g = \tilde{h}$ )  
or  $h = 0$  (then  $g = h_{\infty}$ ).  $\lrcorner$

(c)  $A$  has no unit  $\Rightarrow \Delta(A) \subset B_{A^*}$ ,  $\Delta(A) \cup \{0\}$  is  $w^*$ -compact

$\Delta(A) \subset B_{A^*}$  by Prop. 22

$\Delta(A) \cup \{0\} = \{ h \in A^* ; \forall x, y \in A : h(x+y) = h(x) + h(y) \}$   
 $\lrcorner$   $w^*$ -closed, so  $w^*$ -compact  $\lrcorner$