Proposition X. 22

$\boxed{A}$ Let A be a unital B-algebra, $e \in A$ be the unit
$h \in \Delta(A)$, i.e. $h: A \to \mathbb{C}$ is a nonzero homomorphism
Then

① $h(e) = 1$

$\ulcorner$ $h(e) = h(e \cdot e) = h(e) \cdot h(e) \quad \Rightarrow h(e) = 1$ or $h(e) = 0$
If $h(e) = 0$, then $\forall x \in A: h(x) = h(e \cdot x) = h(e) \cdot h(x) = 0 \cdot h(x) = 0$ $\lrcorner$

② $x \in G(A) \Rightarrow h(x) \neq 0$

$\ulcorner$ $1 = h(e) = h(x \cdot x^{-1}) = h(x) \cdot h(x^{-1})$. $\lrcorner$

③ $\ker h$ is a maximal ideal

$\ulcorner$ As h is a nonzero homomorphism, $\ker h$ is an ideal.
As h is linear, $\ker h$ is of codimension 1, so it is maximal. $\lrcorner$

④ $\forall x \in A: h(x) \in \sigma(x)$

$\ulcorner$ $h(h(x)e - x) = h(x)h(e) - h(x) \overset{①}{=} 0$
$\overset{②}{\Rightarrow} h(x)e - x \notin G(A) \Rightarrow h(x) \in \sigma(x)$ $\lrcorner$

⑤ $\|h\| = 1$

$\ulcorner$ By ④ $|h(x)| \leq r(x) \leq \|x\| \Rightarrow \|h\| \leq 1$
By ① $h(e) = 1 \Rightarrow \|h\| \geq 1$ $\lrcorner$

$\boxed{B}$ Let A be a Banach algebra (unital or not) and $h \in \Delta(A)$.

① $\exists! \ \tilde{h} \in \Delta(A^+)$ s.t. $\tilde{h}(a,0) = h(a)$ for $a \in A$

⌈ uniqueness: by $\boxed{A}$① necessarily $\tilde{h}(0,1) = 1$, so $\tilde{h}(a,\lambda) = h(a) + \lambda$ for $(a,\lambda) \in A^+$

Existence: Define $\tilde{h}(a,\lambda) = h(a) + \lambda$, $(a,\lambda) \in A^+$

- Clearly $\tilde{h}$ is linear
- $\tilde{h}((a,\lambda)(b,\mu)) = \tilde{h}((ab + \lambda b + \mu a, \lambda\mu)) =$
  $= h(ab + \lambda b + \mu a) + \lambda\mu = h(a)h(b) + \lambda h(b) + \mu h(a) + \lambda\mu$
  $= (h(a) + \lambda)(h(b) + \mu) = \tilde{h}(a,\lambda)\,\tilde{h}(b,\mu)$.
- $\tilde{h} \neq 0$ as $\tilde{h}(0,1) = 1 \neq 0$ ⌋

② $\|h\| \leq 1$

⌈ Consider $\tilde{h}$ given by ①. By $\boxed{A}$⑤ we have $\|\tilde{h}\| = 1$. So, $\|h\| \leq 1$ $(h(a) = \tilde{h}(a,0))$. ⌋

③ $h(x) \in \sigma(x)$ for $x \in A$

⌈ A unital ... by $\boxed{A}$③

A not unital $\Rightarrow h(x) = \tilde{h}(x,0) \overset{\boxed{A}③}{\in} \sigma_{A^+}(x,0) = \sigma_A(x)$ ⌋