Proposition X.17

Let $\varphi : [a,b] \to \mathbb{C}$ be a cts piecewise C^1-curve.

i.e. • φ is cts
• $\exists$ $a = t_0 < t_1 < \dots < t_k = b$ s.t.
$\forall j \in \{1, \dots, k\}$ φ' is cts on (t_{j-1}, t_j)
and $\lim_{t \to t_{j-1}+} \varphi'(t)$, $\lim_{t \to t_j -} \varphi'(t)$ exist (finite)

Let $f : \langle \varphi \rangle \to X$ be continuous ($\langle \varphi \rangle = \varphi([a,b])$, X is a Banach space)

Then $\int_\varphi f = \int_a^b f(\varphi(t))\varphi'(t)\,dt$ exists in the Bochner sense

Indeed, let $g(t) := f(\varphi(t))\varphi'(t)$. Then g is cts on $[0,1] \setminus \{t_0, \dots, t_k\}$, so

$g([0,1] \setminus \{t_0, \dots, t_k\})$ is separable (as a cts image of a separable space)

Further, it is cts, so Borel measurable, thus it is strongly measurable by Pettis thm (Theorem VIII.5)

Finally, $f(\varphi([0,1]))$ is compact, hence bdd subset of X
φ' is bdd on each (t_{j-1}, t_j), so it is bdd

It follows that g is bdd, hence $\int_a^b \|g\| < \infty$.

Hence, g is Bochner-integrable. (Theorem VIII.8)

__Remarks__: • $x = \int_\varphi f \iff \forall x^* \in X^*: x^*(x) = \int_\varphi x^* \circ f$

$\ulcorner \Rightarrow$: $x = \int_\varphi f \Rightarrow x = (B)\text{-}\int_a^b f(\varphi(t))\varphi'(t)\,dt$

$\overset{\text{Prop VIII.12}}{\Longrightarrow} \forall x^* \in X^*: x^*(x) = \underbrace{(B)}_{\text{Lebesgue integral}}-\int_a^b \underbrace{x^*(f(\varphi(t))\varphi'(t))}_{= x^*(f(\varphi(t)))\cdot\varphi'(t)}\,dt$

$= \int_\varphi x^* \circ \varphi$

$\Leftarrow$: Assume $\forall x^* \in X^*: x^*(x) = \int_\varphi x^* \circ f$

Let $y = \int_\varphi f$ (exists by Prop. 17)

By "$\Rightarrow$" we deduce that $\forall x^* \in X^*: x^*(y) = x^*(x)$.

Thus $y = x$ (by a consequence to H–B thm ... apply Corollary II.7 to $y-x$) and that's it. $\lrcorner$

• One may use Riemann integral instead of Bochner integral:

$g: [a,b] \to X$, $x \in X$

$x = (R)\int_a^b g \overset{\text{dt}}{\equiv} \forall \varepsilon > 0\ \exists \delta > 0\ \forall a = t_0 < t_1 < \dots < t_k = b$:

$\max_{1 \le j \le k}(t_j - t_{j-1}) < \delta \Rightarrow$

$\forall u_1 \in [t_0, t_1], \dots, u_k \in [t_{k-1}, t_k]$:

$\|x - \sum_{j=1}^{k} g(u_j)(t_j - t_{j-1})\| < \varepsilon$

How to prove that $(R)\int_a^b f(\varphi(t))\varphi'(t)\,dt$ exists:

__Notation__: $D: a = t_0 < t_1 < \dots < t_k = b$ partition, $\nu(D) = \max_{1 \le j \le k}(t_j - t_{j-1})$

$(s_j)_{j=1}^k$ are tags for D if $s_j \in [t_{j-1}, t_j]$ for each j

$S(D, (s_j)_{j=1}^k) = \sum_{j=1}^{k} g(u_j)(t_j - t_{j-1})$

Step 1: $g : [\alpha, \beta] \to X$ cts $\Rightarrow$ (R) $\int_\alpha^\beta g$ exists

⌈ Similarly as in the scalar case: g is uniformly cts

so, given $\varepsilon > 0$, there is $\delta > 0$ s.t. $|s-t| < \delta \Rightarrow \|g(s) - g(t)\| < \varepsilon$

If D is a partition with $\nu(D) < \delta$, (s_j), (u_j) two sets of tags, then $\|S(D, (s_j)) - S(D, (u_j))\| < \varepsilon(\beta - \alpha)$

More generally, if D is a partition with $\nu(D) < \delta$, (s_j) its tags
D' is a refinement of D, (u_j) its tags
$\Rightarrow \|S(D, (s_j)) - S(D', (u_j))\| < \varepsilon(\beta - \alpha)$

Now, let D_1, D_2 be two partitions with $\nu(D_j) < \delta$
Let (s_j), (u_j) be their tags
Let D be a common refinement, (v_j) its tags
Then $\|S(D_1, (s_j)) - S(D_2, (u_j))\| \leq \|S(D_1, (s_j)) - S(D, (v_j))\|$
$+ \|S(D, (v_j)) - S(D_2, (u_j))\| < 2\varepsilon(\beta - \alpha)$.

Hence: $\forall \varepsilon > 0\ \exists \delta$: $\nu(D_1), \nu(D_2) < \delta \Rightarrow \|S(D_1, (s_j)) - S(D_2, (t_j))\|$
$< 2\varepsilon(\beta - \alpha)$

So, by completeness of X we get the existence of integral ⌋

Step 2: (R) $\int_\alpha^\beta g$ exists, $h = g$ on (α, β) $\Rightarrow$ (R) $\int_\alpha^\beta h$ exists and $=$ (R) $\int_\alpha^\beta g$

⌈ g, h are bounded. Assume $\|h\| \leq M$, $\|g\| \leq M$. $x =$ (R) $\int_\alpha^\beta g$
$\varepsilon > 0$ .... $\exists \delta > 0$ s.t. .... (as in the definition)

let D be a partition with $\nu(D) < \delta$. Let (u_j) be some tags
Then $\|S_h(D, (u_j)) - x\| \leq \underbrace{\|S_h(D, (u_j)) - S_g(D, (u_j))\|}_{\leq 4M\delta \text{ (from the first and the last intervals)}} + \underbrace{\|S_g(D, (u_j)) - x\|}_{< \varepsilon}$

⌋

Step 3 : $(R)\int_\alpha^\beta g$, $(R)\int_\beta^{\varphi} g$ exist $\Rightarrow$ $(R)\int_\alpha^{\varphi} g$ exists and is equal to $(R)\int_\alpha^\beta g + (R)\int_\beta^{\varphi} g$

Γ $x = (R)\int_\alpha^\beta g$, $y = (R)\int_\beta^{\varphi} g$. $\varepsilon > 0 \ldots \exists \delta > 0 \ldots$ (common for the two integrals)

D partition of $[\alpha, \varphi]$, $\nu(D) < \delta$, (u_j) tags

$D' :=$ refinement of D by adding β $\quad$ (v_j) --- the same tags with one added to the relevant interval

D_1 --- the respective partition of $[\alpha, \beta]$ $\quad$ (v_j^1)

D_2 --- the respective partition of $[\beta, \varphi]$ $\quad$ (v_j^2) $\Big\}$ the respective tags.

Then $\| S(D, (u_j)) - (x+y) \| \le \underbrace{\| S(D, (u_j)) - S(D', (v_j)) \|}_{\le 2M\delta \text{ where } (\|g\| \le M)}$

$+ \| \underbrace{S(D', (v_j))}_{= S(D_1, (v_j^1)) + S(D_2, (v_j^2))} - (x+y) \| \le 2M\delta + \underbrace{\| S(D_1, (v_j^1)) - x \|}_{< \varepsilon} + \underbrace{\| S(D_2, (v_j^2)) - y \|}_{< \varepsilon}$

$< 2M\delta + 2\varepsilon$. $\lrcorner$

Conclusion: By Steps 1, 2, 3 we see that $(R)\int_a^b f(\varphi(t))\varphi'(t)\,dt$ exists.

- $x = (R)\int_\alpha^\beta g \Rightarrow \forall x^* \in X^*$ $x^*(x) = (R)\int_\alpha^\beta x^* \circ g$

Γ $| S_{x^*\circ g}(D, (u_j)) - x^*(x) | = | x^*(S_g(D, (u_j)) - x) | \le \|x^*\| \, \| S_g(D, (u_j)) - x \|$ $\lrcorner$